The 4 Postulate in Density-Operator Formalism:

Postulate 1: An isolated physical system is completely described by its density operator, ρ, which is a trace-one, positive operator in a Hilbert space.

$|\psi\rangle$ $\quad |\psi\rangle\langle\psi|$

① $|\psi\rangle$

② p_i, $|\psi_i\rangle$

Moreover, if a system is at state ρ_i w.p. p_i, then the system is described by $\sum_i p_i \cdot \rho_i$

① $1 \cdot |\psi\rangle\langle\psi|$

② $\sum_i p_i \cdot |\psi_i\rangle\langle\psi_i|$

w.p. p_i, ρ_i

Possible: $\rho_i = \sum_j p_j^{(i)} \rho_j^{(i)}$

$\sum_k p^{(i,j,k)} \rho^{(ijk)}$

Postulate 2: The evolution of a closed quantum system is described by unitary operators (U). Notation-wise:

$$\rho \xrightarrow{U} U\rho U^\dagger$$

$$\begin{array}{ccc} \rightarrow\ |\psi\rangle & \xrightarrow{U} & U|\psi\rangle \quad \text{pure} \\ \downarrow \text{d.o.} & & \downarrow \\ \rightarrow\ \rho = |\psi\rangle\langle\psi| & \xrightarrow{U} & U|\psi\rangle (U|\psi\rangle)^\dagger \end{array}$$

$$\rho = |\psi\rangle\langle\psi| = |\psi\rangle (|\psi\rangle)^\dagger$$

$$U|\psi\rangle (U|\psi\rangle)^\dagger = U|\psi\rangle\langle\psi| U^\dagger$$

$$\rho = \sum_i p_i \rho_i$$

$$\Big\downarrow U$$

$$U\rho U^\dagger = \sum_i p_i \cdot U\rho_i U^\dagger$$

Postulate 3: Measurement (Born's Rule)

Quantum measurements are described by a collection of matrices $\{M_m\}_m$ satisfying the completeness condition:

$$\sum_m M_m^\dagger M_m = \mathbb{I}$$

If a state ρ is measured

In a pure state $|\psi\rangle$:

Observe M-outcome m. w.p.

$$p_m = \langle\psi| M_m^\dagger M_m |\psi\rangle = tr[\langle\psi| M_m^\dagger M_m |\psi\rangle] = tr[|\psi\rangle\langle\psi| M_m^\dagger M_m]$$

with post-M state being:

$$\frac{M_m|\psi\rangle}{\| M_m|\psi\rangle \|} = \frac{M_m|\psi\rangle}{\sqrt{\langle\psi| M_m^\dagger M_m|\psi\rangle}}$$

$\rho = |\psi\rangle\langle\psi|$

$$tr[A\cdot B\cdot C] = tr[C\cdot A\cdot B] = tr[B\cdot C\cdot A]$$

observe m. w.p. $p_m = tr[\rho \cdot M_m^\dagger M_m]$

$$= \mathrm{tr}[\underbrace{M_m \rho M_m^\dagger}] = \mathrm{tr}[M_m^\dagger M_m \rho]$$

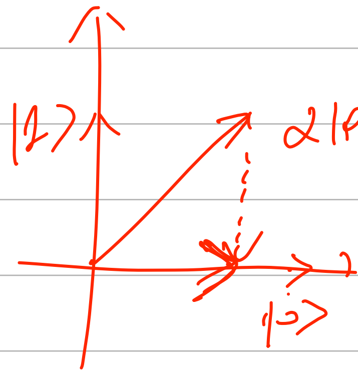
with Post-M state being:

$$\rho_m = \frac{M_m|\psi\rangle}{\sqrt{\langle\psi|M_m^\dagger M_m|\psi\rangle}} \cdot \left(\frac{M_m|\psi\rangle}{\sqrt{\cdots}}\right)^\dagger$$

$$= \frac{M_m|\psi\rangle}{\sqrt{\cdots}} \cdot \frac{\langle\psi| \cdot M_m^\dagger}{\sqrt{\cdots}} = \frac{M_m|\psi\rangle\langle\psi|M_m^\dagger}{\underbrace{\langle\psi|M_m^\dagger M_m|\psi\rangle}_{p_m}}$$

$$= \frac{M_m|\psi\rangle\langle\psi|M_m^\dagger}{\mathrm{tr}[M_m|\psi\rangle\langle\psi|M_m^\dagger]}$$

$$= \frac{M_m \rho M_m^\dagger}{\mathrm{tr}[M_m \rho M_m^\dagger]}$$



Postulate 4: The state space of a composite physical system is the tensor product of the spaces of the component physical systems.

$\rho_1 = |\psi\rangle\langle\psi|$ $\qquad$ $\rho_2 = |\phi\rangle\langle\phi|$

$|\psi\rangle$ $\qquad$ $|\phi\rangle$

H_1 $\quad$ H_2 $\quad$ ρ_1 $\quad$ ρ_2 $\quad$ $\rho_3 = (|\psi\rangle|\phi\rangle)(\langle\psi|\langle\phi|)$

$$H_1 \otimes H_2 := \underline{[\rho_1][\rho_2]} \qquad \underline{|\psi\rangle \otimes |\phi\rangle}$$

$$= |\psi\rangle|\phi\rangle = |\psi, \phi\rangle$$

$$(U_1 \otimes I_2)\, |\psi\rangle \otimes |\phi\rangle = U_1|\psi\rangle \otimes I|\phi\rangle$$

$$\boxed{\text{Claim 1: } \rho_3 = \rho_1 \otimes \rho_2}$$

<u>proof for claim 1:</u>

$$\rho_3 = (\underset{n\times 1}{|\psi\rangle}\, \underset{n\times 1}{|\phi\rangle})\, (\underset{1\times n}{\langle\psi|}\, \underset{1\times n}{\langle\phi|})$$

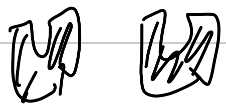
$$\rho_1 \otimes \rho_2 = (|\psi\rangle\langle\psi|) \otimes (|\phi\rangle\langle\phi|)$$

Property of kronecker product

$$(A \otimes B)(C \otimes D) = (A\cdot C) \otimes (B \cdot D)$$ (as long as Matrix demension allows you to do so)

- Entangled states vs. non-entangled states

$$\underbrace{\frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)} \qquad \underbrace{|\phi\rangle \otimes |\psi\rangle}$$



- (non-pure) Mixed states vs. pure states

w.p. p_i, have $|\psi_i\rangle$ $\qquad$ $|\psi\rangle$

$\left\{\begin{array}{l} \text{ensemble.} \\ \text{distribution} \\ \text{Mixture} \end{array}\right.$ of pure states

$$\frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right); \quad = \rho_1$$

Question 1: Is it a mixture of the dist.? $\qquad$ No

$$\left[\begin{array}{lll} \text{w.p. } \frac{1}{2}, & & |00\rangle \\ \frac{1}{2}, & & |11\rangle \end{array}\right] \rho_2$$

$$\rho_2 = \frac{1}{2}|00\rangle\langle 00| + \frac{1}{2}|11\rangle\langle 11|$$

$$\rho_1 = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) \cdot \frac{1}{\sqrt{2}}\left(\langle 00| + \langle 11|\right)$$

$$= \frac{1}{2}\big(|00\rangle\langle 00| + |11\rangle\langle 11| + |00\rangle\langle 11| + |11\rangle\langle 00|\big)$$

state 1: $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ $\leftarrow$

state 2: $\begin{cases} \text{w.p. } \frac{1}{2}, \text{ have } |0\rangle & 0 \\ \text{w.p. } \frac{1}{2}, \text{ have } |1\rangle & 1 \end{cases}$ $\leftarrow$

$$\rho_1 = \frac{1}{2}\big(|0\rangle\langle 0| + |1\rangle\langle 1| + |0\rangle\langle 1| + |1\rangle\langle 0|\big)$$

$$\rho_2 = \frac{1}{2}\big(|0\rangle\langle 0| + |1\rangle\langle 1|\big)$$

Thm: (Criterion to determine if a state is Mixed or pure). $\mathrm{tr}[\rho] = 1$

Let ρ be a density operator.

① $\mathrm{tr}[\rho^2] \le 1$

② $\mathrm{tr}[\rho^2] = 1$ iff ρ is a pure state

$tr[\rho_1^2] = 1$

$tr[\rho_2^2] = tr[\cdots\cdots]$

$$\downarrow$$

$$tr\left[\left(\frac{1}{2}(|0\rangle\langle 0| + \underbrace{|1\rangle\langle 1|}\,)\right)\left(\frac{1}{2}(|0\rangle\langle 0| + \underbrace{|1\rangle\langle 1|}\,)\right)\right]$$

$$= tr\left[\frac{1}{4}\left(|0\rangle\underbrace{\langle 0|0\rangle}_{1}\langle 0| + |1\rangle\langle 1| + |0\rangle\langle 0| \cdot |1\rangle\langle 1| + |1\rangle\langle 1| \cdot |0\rangle\langle 0|\right)\right]$$

$$= tr\left[\frac{1}{4}(|0\rangle\langle 0| + |1\rangle\langle 1|)\right]$$

$$= \cancel{\frac{1}{2}}\ \frac{1}{4}\left(\underbrace{tr[|0\rangle\langle 0|]} + \cancel{tr}[|1\rangle\langle 1|]\right)$$

$$= tr[\langle 0|0\rangle]$$

$$= 1$$

$$= \frac{1}{2}$$

$$\begin{cases} \text{w.p. } \frac{1}{2} \ — \ |0\rangle < \begin{matrix} \frac{1}{2} & |+\rangle \\ \frac{1}{2} & |-\rangle \end{matrix} \\ \text{w.p. } \frac{1}{2} \ — \ |1\rangle < \begin{matrix} \frac{1}{2} & |+\rangle \\ \frac{1}{2} & |-\rangle \end{matrix} \end{cases}$$

$\left(\frac{1}{4} + \frac{1}{4}\right)$ $|+\rangle$

$\left(\frac{1}{4} + \frac{1}{4}\right)$ $|-\rangle$

Global phase.

$$|\psi\rangle = -|\psi\rangle$$

$$= e^{i\theta}|\psi\rangle \qquad \theta \in [0, 2\pi)$$

$$\rho_1 = |\psi\rangle\langle\psi|$$

$$\rho_2 = e^{i\theta}|\psi\rangle \left(e^{i\theta}|\psi\rangle\right)^\dagger$$

$$= e^{i\theta}|\psi\rangle \cdot e^{-i\theta}\langle\psi|$$

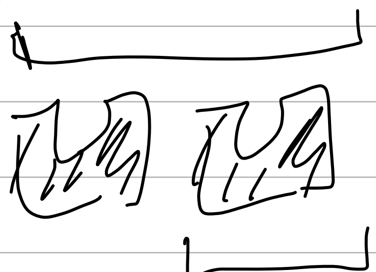
$$= e^{i\theta} \cdot e^{-i\theta}|\psi\rangle\langle\psi|$$

$$= e^{i\theta - i\theta}|\psi\rangle\langle\psi|$$

$$= e^{0}|\psi\rangle\langle\psi|$$

$$= |\psi\rangle\langle\psi|$$

Reduced Density Operator



Alice Bob

$$\frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

Def : (Partial Trace)

$$\mathrm{tr}_A[\rho_{AB}] := \sum_{j=1}^{d} \overbrace{(\langle e_j|_A \otimes \mathbb{I}_B)}^{\text{input}} \rho_{AB} (|e_j\rangle_A \otimes \mathbb{I}_B)$$

└ "trace out" A system
delete / remove

where $\{|e_j\rangle_A\}_{j=1}^{d}$ is an orthonormal basis of system A.

Remark:

[Nielsen - Chuang] def partial trace in (Section 2.4.3) in a different way.

① equivalent to the above

② "bad" : it doesn't tell you how to calculate partial trace of given states

Partial trace completely describes ~~the~~ the view of a partial system in a larger

└ system.

$$\rho_{AB} = \overbrace{\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)}^{|\psi\rangle} \cdot \overbrace{\frac{1}{\sqrt{2}}(\langle 00| + \langle 11|)}^{\langle\psi|}$$

Alice Bob

$\{|0\rangle_A , |1\rangle_A\}$

$$tr_A[\rho_{AB}] = \underbrace{(\langle 0|_A \otimes I_B)\overbrace{\rho_{AB}}^{|\psi\rangle\langle\psi|}(|0\rangle_A \otimes I_B)}$$

$$+ \underbrace{\langle 1|_A \otimes I_B)} \rho_{AB} (|1\rangle_A \otimes I_B)$$

$$= \left(\frac{1}{\sqrt{2}}|0\rangle_B\right)\left(\frac{1}{\sqrt{2}}\langle 0|_B\right) + \left(\frac{1}{\sqrt{2}}|1\rangle_B\right)\left(\frac{1}{\sqrt{2}}\langle 1|_B\right).$$

$$= \frac{1}{2}(|0\rangle\langle 0|_B + |1\rangle\langle 1|_B)$$

$$\underbrace{(\langle 0|_A \otimes I)} \left(\frac{1}{\sqrt{2}}(\underbrace{|00\rangle}_{1} + \underbrace{|11\rangle}_{7})\right)$$

$$= \frac{1}{\sqrt{2}}\underbrace{(\langle 0|_A \otimes I_B)(|00\rangle)} + \frac{1}{\sqrt{2}}(\langle 0|_A \otimes I_B)(|11\rangle)$$

$$\underline{(A \otimes B)\cdot(C \otimes D) = (A \cdot C) \otimes (B \cdot D)}$$

$$\underbrace{(\langle 0|_A |0\rangle_A)}_{1} \otimes \underbrace{(I_B \cdot |0\rangle_B)}_{|0\rangle_B}$$

$= \frac{1}{\sqrt{2}} \cdot |0\rangle_B + 0$

If A measures under comp. basis.

w.p. $\frac{1}{2}$. $|00\rangle$ $\quad |0\rangle_B$

$\frac{1}{2}$. $|11\rangle$ $\quad |1\rangle_B$

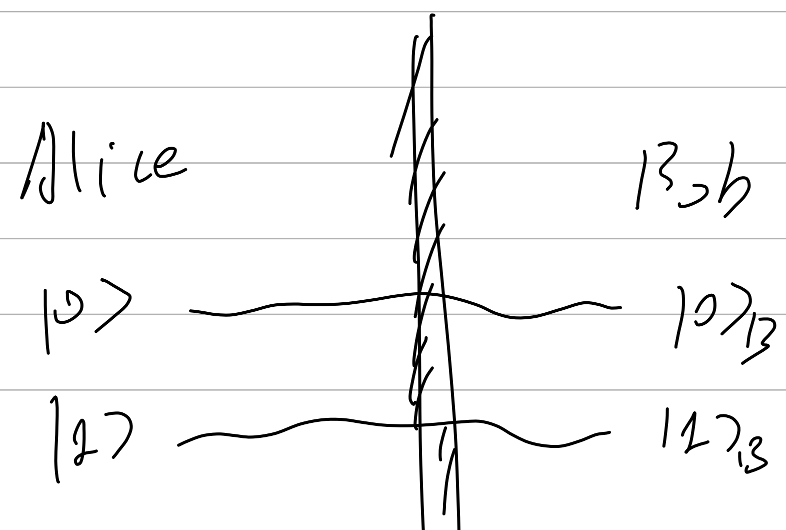
If A measures under Hadamard basis.

$|+\rangle, |-\rangle$

w.p. $\frac{1}{2}$ Bob observes $|0\rangle_B$

$\frac{1}{2}$ - - - - $|1\rangle_B$

$\rightarrow U_A^0$



U_A^1

$$\begin{cases} \left(U_A^0 \otimes I_B\right) \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) = |00\rangle_{AB} \\ \left(U_A^1 \otimes I_B\right) \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) = |11\rangle_{AB} \end{cases}$$