Recall:

EPR Paradox (by Einstein, Podolsky, Rosen)

- Realism
- locality.

$\Big\}$ no concrete reasons
more like a belife.
so far so good with "pre-quantum" physics.

EPR's thoughts:

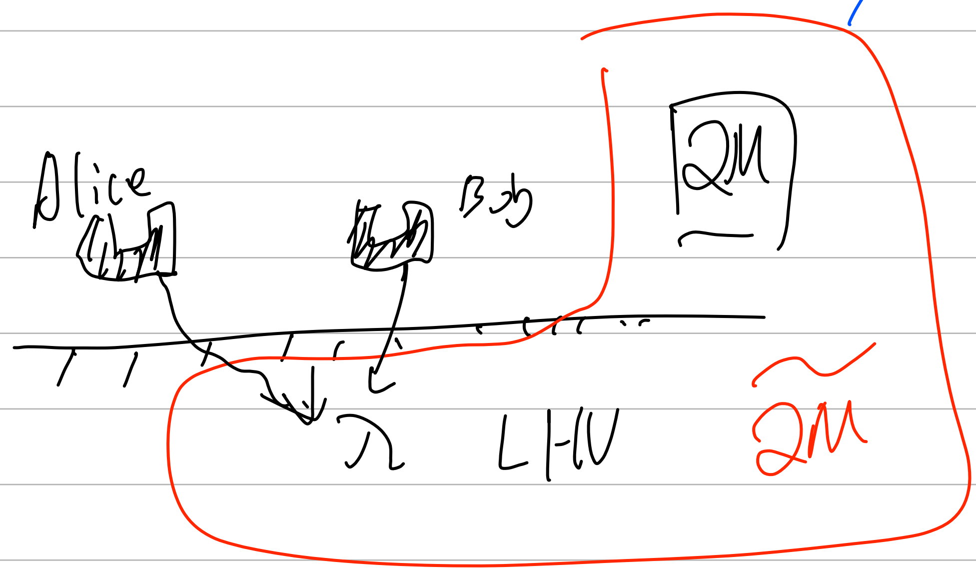
- QM is not "complete". Need to be extended.

(recall Newton's mechanics v.s. special Relativity).

- There should be a Local Hidden Variable (LHV) theory.

$\boxed{\begin{matrix}\text{Realism} \\ \text{Locality}\end{matrix}} \sim \text{LHV} \; \cancel{\sim} \; \boxed{\text{QM}}$

by John Bell.



Is LHV real or QM real?

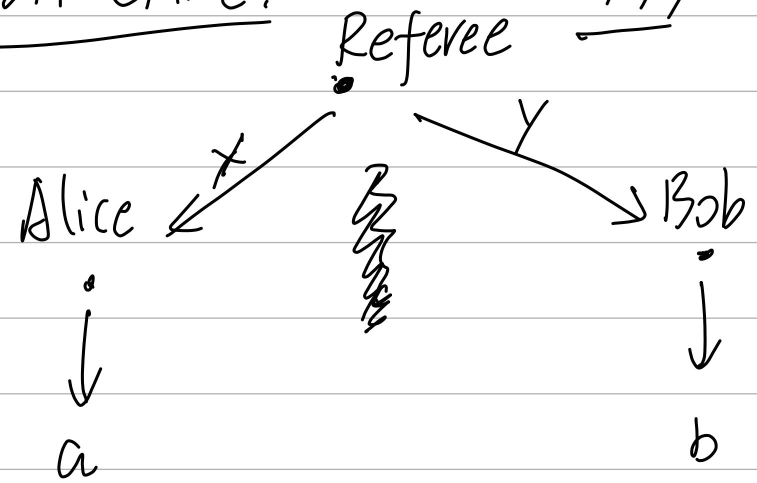
- It went unresolved for ~30 years.

Year 1964, John Bell.

No Local Hidden Varicible theory can be compatible with QM.

1970, by Clauser, Horne, Shimony, Holt (CHSH).

CHSH Game: $x, y \leftarrow \{0,1\}$



Win Condition: $a \oplus b = x \wedge y$

X	Y	Win condition
0	0	$a=b$
0	1	$a=b$
1	0	$a=b$
1	1	$a \neq b$

Prove:

① There exists an upper bound for the best winning prob. of LHV.

② provide a QM strategy, so. that
Prob.[Win] > upper bound in ①

①+② ⇒ LHV was wrong.

Task ②:

Case 1: deterministic strategy:

$$\begin{cases} f_{Alice}(x) = a \\ f_{Bob}(y) = b \end{cases}$$

x	a
0	·
1	·

Alice

Case 2: Probabilistic strategy:

$f_{Alice}(x; r_A) = a$

$r_A \in \{0,1\}^n$

$f_{Bob}(y; r_B) = b$.

x	r_A	a
0	....	
0	...	
0	...	...
0	...	...

2^n

Assume the best strategy is

$$\begin{cases} \tilde{f}^{\,r_A}_{Alice}(x) = a \\ \tilde{f}^{\,r_B}_{Bob}(y) = b \end{cases}$$

x	a
0	
1	

Alice

Claim 1: $\max\{\Pr[Win]\} = \frac{3}{4}$

Proof: try all 16 possible combinations of Alice & Bob's strategies

Case 3: LHV

$$f_{Alice}(x;\ \cancel{r_A}\ \lambda) = a$$

$$f_{Bob}(y;\ \lambda) = b$$

$$\cancel{r_B}$$

$\lambda \leftarrow \{0,1\}^n$

$\Downarrow$

$$\begin{cases} f_{Alice}(x;\lambda) = a \\ f_{Bob}(y;\lambda) = b \end{cases}$$

Claim 2: Even in Case 3:

$$\max\{\Pr[\text{Win}]\} \leq \frac{3}{4}$$

Restate the CHSH game with LHV strategies:

① Before game starts, $\lambda \leftarrow$ Distribution
└ some dist. that maximizes $\Pr[\text{Win}]$

② Referee samples and sends (x, y)

③ Alice outputs $a = \tilde{f}_{Alice}(x; \lambda)$.

④ Bob outputs $b = \tilde{f}_{Bob}(y; \lambda)$.

⑤ Check if $a \oplus b = x \wedge y$.

Law of total Probability, (LTP)

$$\Pr[A] = \Pr[A \wedge B] + \Pr[A \wedge \neg B]$$

$$\Pr[A \wedge B] = \Pr[B] \cdot \Pr[A|B]$$

Proof of Claim 2:

$$\Pr[\text{Win}] \overset{\text{by LTP}}{=} \sum_{\lambda \in \{0,1\}^n} \Pr[\text{Win} \wedge (\Lambda = \lambda)]$$

$$\overset{\text{chain rule}}{=} \sum_{\lambda} \Pr[\Lambda = \lambda] \cdot \Pr[\text{Win} \mid \lambda]$$

by Claim 1:

$$\leq \underbrace{\left(\sum_{\lambda} \Pr[\Lambda = \lambda]\right)}_{=1} \cdot \frac{3}{4}$$

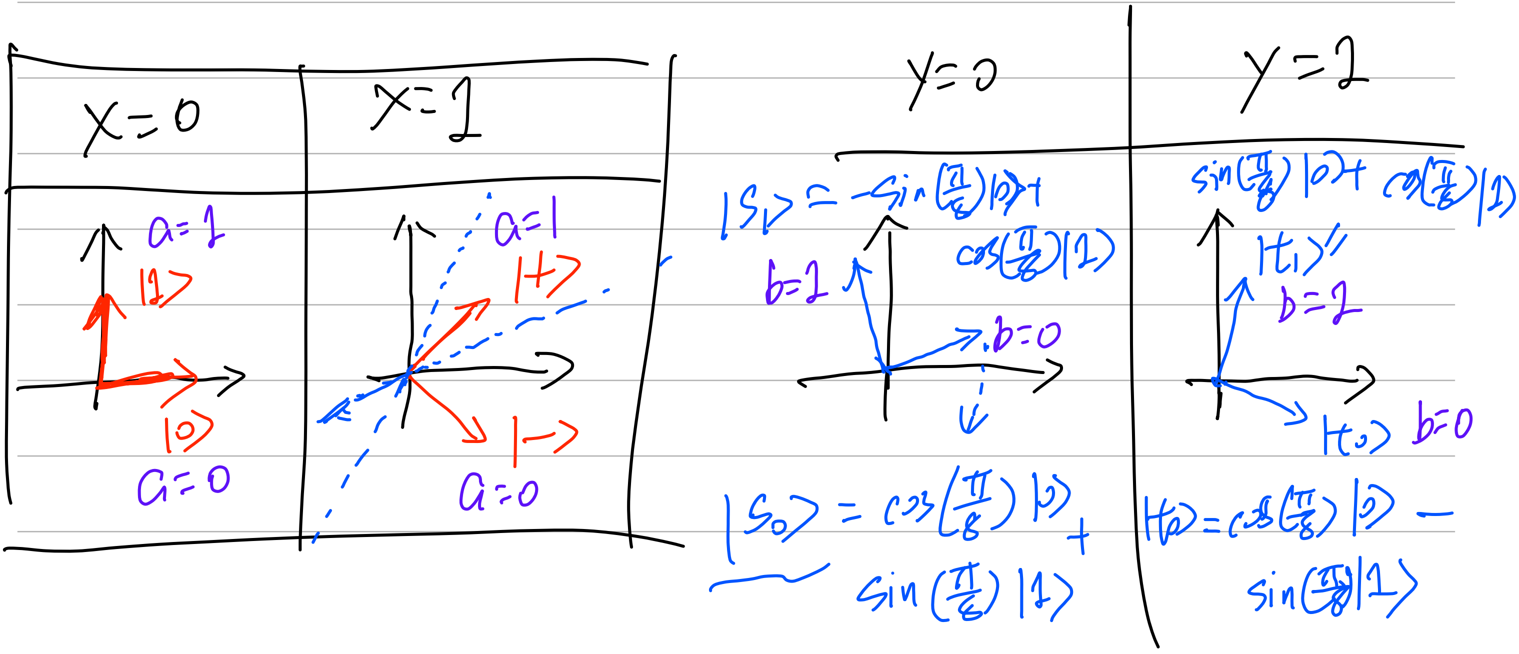
$$= 1 \cdot \frac{3}{4} = \frac{3}{4} \qquad \square$$

Task 2:

$$\underbrace{\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)}$$

Alice ← qubit, qubit → Bob

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$



Claim 3: For all possible (x, y) pairs,

$$\Pr[\text{Win}] = \cos^2\left(\frac{\pi}{8}\right) \approx 0.8535\ldots$$

Proof:

case $(x=1, y=1)$

Lem: It doesn't matter who measures first

proof: We'll see it when ~~talking~~ about density-operator formalism

Alice's measurement in $x=1$:

$$\begin{cases} M_0 = |+\rangle\langle+| & \Rightarrow M_0^\dagger M_0 = |+\rangle\langle+| \\ M_1 = \mathbb{I} - |+\rangle\langle+| = |-\rangle\langle-| & \Rightarrow M_1^\dagger M_1 = |-\rangle\langle-| \end{cases}$$

observe 0 w.p. $\underline{\langle\psi| M_0^\dagger M_0 |\psi\rangle}$

$$\begin{cases} \tilde{M}_0 = |+\rangle\langle+|_{\text{Alice}} \otimes \mathbb{I}_{\text{Bob}} \\ \tilde{M}_1 = |-\rangle\langle-|_{\text{Alice}} \otimes \mathbb{I}_{\text{Bob}} \end{cases}$$

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

requires calculation

Alice observes 0 w.p. $\langle\psi| \tilde{M}_0^\dagger \tilde{M}_0 |\psi\rangle = \frac{1}{2}$

$\rightarrow |+\rangle_{Alice} \Rightarrow$ Overall state $|+\rangle_{Alice} \otimes |+\rangle_{Bob}$

requires calculation.

a

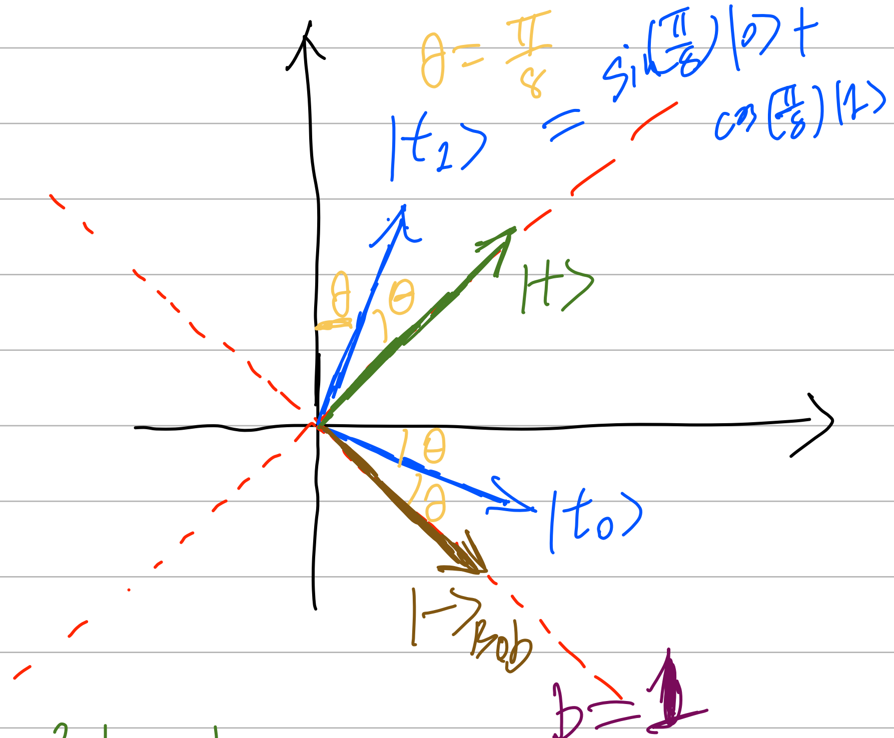
$$\left[\frac{\tilde{M}_0 |\psi\rangle_{Alice,Bob}}{\sqrt{\langle\psi|\tilde{M}_0^{\dagger}\tilde{M}_0|\psi\rangle}} = |+\rangle|+\rangle \right.$$

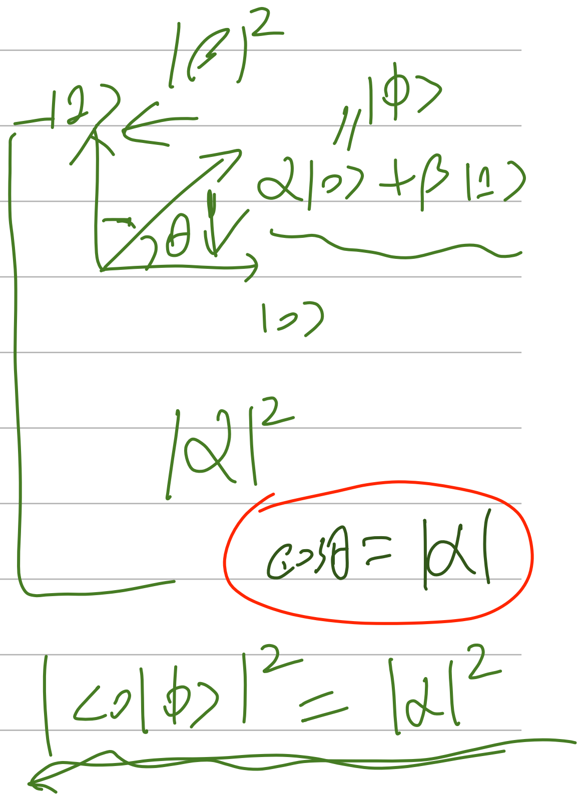
Alice observes 1 w.p. $\frac{1}{2}$,

and the overall state collapses to $|-\rangle_{Alice} \otimes |-\rangle_{Bob}$

Bob's strategy:

When $|+\rangle_{Bob}$:





Bob observes $|t_1\rangle$ w.p.

$$|\langle t_1|+\rangle|^2 = \cos^2(\theta) = \cos^2(\frac{\pi}{8})$$

when $|-\rangle_{Bob}$:

— symmetric of $|+\rangle_{Bob}$.

Bob observes $|t_0\rangle^{b=0}$ w.p.

$$|\langle t_0|-\rangle|^2 = \cos^2(\theta) = \cos^2(\tfrac{\pi}{8})$$

Nobel Prize in Physics, 2022

Alain Aspect, John Clauser, Zeilinger

"for experiments with entangled photons, establishing the violation of Bell inequality and pioneering quantum information science!"

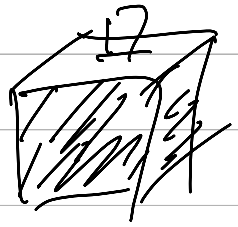
Density Operators

Motivations.

1. no math way to talk about a single person's view for

his/her share of an EPR pair

$$2: \begin{cases} |\psi\rangle = \alpha|0\rangle + \beta|1\rangle \\ |\alpha|^2 + |\beta|^2 = 1 \end{cases}$$



$$\underbrace{\begin{cases} \text{w.p. } \underline{p_1}, & \underline{|\psi_1\rangle} \\ \text{w.p. } \underline{p_2}, & \underline{|\psi_2\rangle} \end{cases}}$$

$$p_1 |\psi_1\rangle\langle\psi_1| + p_2 |\psi_2\rangle\langle\psi_2|$$

If you have a dist. over pure states.

$$\{\text{w.p. } p_i, \text{ have } |\psi_i\rangle\}_{i=1}^{n}$$

the density operator is

$$\rho = \sum_{i=1}^{n} p_i \cdot |\psi_i\rangle\langle\psi_i|$$

$$\begin{cases} ① \ \sum_{i=1}^{n} p_i = 1 \\ ② \ \| |\psi_i\rangle \| = 1 \end{cases}$$

Thm: A matrix ρ is a density operator (of some dist. over pure states $\{p_i, |\psi_i\rangle\}$)

if and only if:

1. (trace condition): $tr[\rho] = 1$
2. (Positivity condition): ρ is a positive operator.

$\Leftrightarrow \forall |\psi\rangle, \ \langle\psi|\rho|\psi\rangle \geq 0$

$\Leftrightarrow$ ρ is positive operator / positive semi-def. matrix.

$\Leftrightarrow$ all eigenvalues of $\rho \geq 0$