Basic Quantum-Exclusive Effects (cont'd)

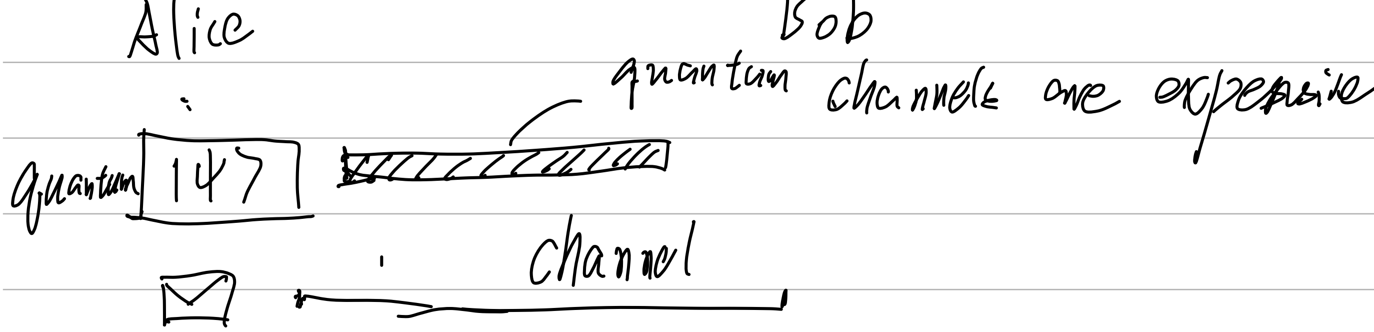
Quantum Teleportation.

Controled-NOT (CNOT)

$$\mathrm{CNOT} ::= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\forall a, b \in \{0,1\}, \quad \mathrm{CNOT}\,|a, b\rangle := |a, b \oplus a\rangle$$

$$= \begin{cases} |0, b\rangle & a=0 \\ |1, \neg b\rangle & a=1 \end{cases}$$



Can we transmit quantum data over classical channels? — NO. (strong evidence)

$\mathrm{Enc}(|\psi\rangle) = \mathrm{bits}_{|\psi\rangle}$

Alice $\xrightarrow{\mathrm{bits}_{|\psi\rangle}}$ Bob

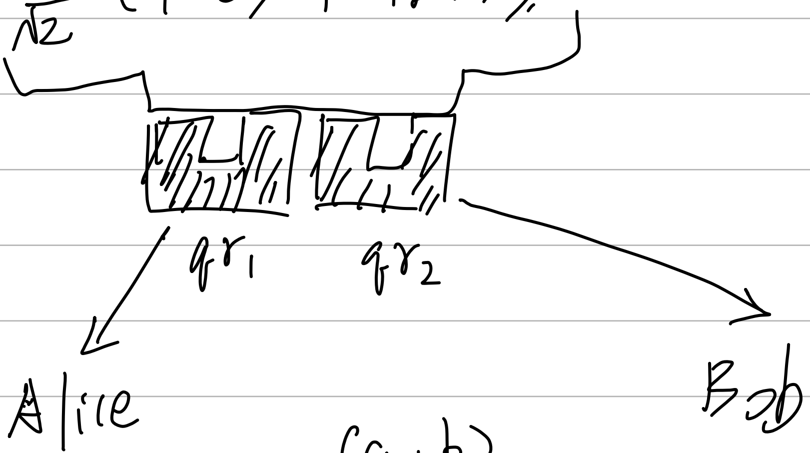
$\mathrm{Dec}(\mathrm{bits}_{|\psi\rangle})$

$\downarrow$

$|\psi\rangle$

EPR pair allows it!

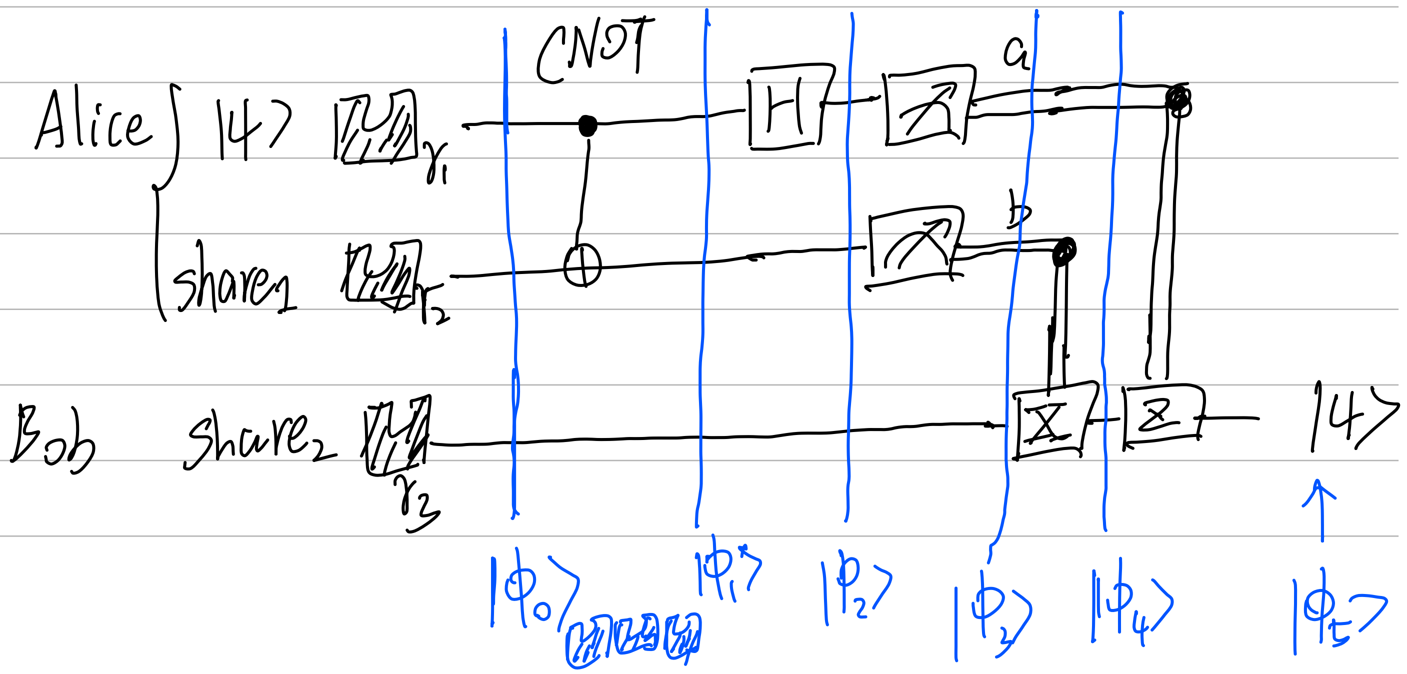
$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$



$|\psi\rangle \xrightarrow[\text{classical}]{(a,\, b)}$

— quantum wire

= classical wire



$$|\phi_0\rangle = |\psi\rangle \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$\hookrightarrow \alpha|0\rangle + \beta|1\rangle \qquad (|\alpha|^2 + |\beta|^2 = 1)$$

$$= (\alpha|0\rangle + \beta|1\rangle) \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$= \frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle)$$

$$|\phi_1\rangle = (CNOT_{r_1 r_2} \otimes \mathbb{I}_{r_3})|\phi_0\rangle_{r_1 r_2 r_3}$$

$$= \frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|011\rangle + \beta|110\rangle + \beta|101\rangle)$$

$$|011\rangle = |0\rangle \otimes |1\rangle \otimes |1\rangle$$

$$|\phi_2\rangle = (H_{r_1} \otimes \mathbb{I}_{r_2 r_3})|\phi_1\rangle_{r_1 r_2 r_3}$$

$$= \frac{1}{\sqrt{2}}(\alpha|+\rangle|00\rangle + \alpha|+\rangle|11\rangle + \beta|-\rangle|10\rangle + \beta|-\rangle|01\rangle)$$

by $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

$$= \frac{1}{2}\{\alpha|000\rangle + \alpha|100\rangle + \alpha|011\rangle + \alpha|111\rangle +$$
$$\beta|010\rangle - \beta|110\rangle + \beta|001\rangle - \beta|101\rangle\} \qquad \begin{pmatrix} a=1 \\ b=0 \end{pmatrix}$$

Measure the (r_1, r_2) Register under stand-basis.

- $(a=0, b=0)$:

$$\overbrace{\alpha|000\rangle}^{r_1 r_2 r_3} + \beta|001\rangle$$

$$= |00\rangle_{r_1 r_2}(\alpha|0\rangle_{r_3} + \beta|1\rangle_{r_3})$$

$$Pr = \left(\frac{1}{2}\right)^2 \cdot \left(|\alpha|^2 + |\beta|^2\right)$$

$\downarrow$ by $|\alpha|^2 + |\beta|^2 = 1$

$$= \frac{1}{4}$$

$\begin{cases} M_m = |00\rangle\langle 00| \\ m = (a=0, b=0) \end{cases}$

Post-M state:

$$\frac{1}{2}\left(\alpha|000\rangle + \beta|001\rangle\right) \quad M_m|\phi_2\rangle$$

$$\frac{M_m|\phi_2\rangle}{\sqrt{\langle\phi_2|M_m^\dagger M_m|\phi_2\rangle}} = \frac{\frac{1}{2}\left(\alpha|000\rangle + \beta|001\rangle\right)}{\sqrt{\frac{1}{4}}}$$

$$= \alpha|000\rangle + \beta|001\rangle$$

$$= |00\rangle_{r_1 r_2} \underbrace{\left(\alpha|0\rangle_{r_3} + \beta|1\rangle_{r_3}\right)}_{|\psi\rangle_{r_3}}$$

$(a=1, b=0)$ $\quad Pr = \frac{1}{4}$

$$\frac{1}{2}\left(\alpha|100\rangle - \beta|101\rangle\right) \quad \text{(sub-normalized)}$$

$$\alpha|100\rangle - \beta|101\rangle := |\phi_2\rangle$$

$$|\phi_3\rangle = \left(I_{r_1 r_2} \otimes Z^a_{r_3}\right)\left(I_{r_1 r_2} \otimes X^b_{r_3}\right)|\phi_2\rangle$$

$$= (\mathbb{I}_{r_1 r_2} \otimes Z_{r_3})\, \mathbb{I}_{r_1 r_2 r_3} (\alpha|100\rangle - \beta|101\rangle)$$

$$= \mathbb{I}_{r_1 r_2} \otimes Z_{r_3} \left[|10\rangle_{r_1 r_2} (\alpha|0\rangle_{r_3} - \beta|1\rangle_{r_3}) \right]$$

$$= |10\rangle_{r_1 r_2} \left[Z_{r_3} (\alpha|0\rangle_{r_3} - \beta|1\rangle_{r_3}) \right]$$

$$= |10\rangle_{r_1 r_2} (\alpha|0\rangle_{r_3} + \beta|1\rangle_{r_3})$$

$$= |10\rangle_{r_1 r_2} |\psi\rangle_{r_3} \qquad \begin{cases} Z|+\rangle = |-\rangle \\ Z|-\rangle = |+\rangle \end{cases}$$

Closing Remarks:

- Violate No cloning?

  No.

- Compressing the commu. cost further?

  No. (Talk more when talking about EPR paradox)

Super Dense Coding (Encoding)

0/1

$|\psi\rangle \in \mathbb{C}^2 \qquad |\psi\rangle = \underline{\alpha}|0\rangle + \underline{\beta}|1\rangle$

$$|\alpha|^2 + |\beta|^2 = 1$$

$$\pi = 3.1415\ldots$$

$$\alpha_\pi = \frac{\pi}{10} = 0.314159265\ldots$$

$$\beta_\pi = 1 - |\alpha|^2 = 1 - \frac{\pi^2}{100}$$

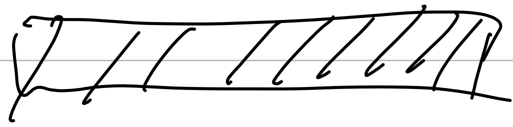
$$|\psi_\pi\rangle = \alpha_\pi|0\rangle + \beta_\pi|1\rangle$$

- Can we encode infinite amount of information in a qubit that allows reliable decoding?
- No!

You cannot encode more than 1 classical bit using 1 qubit

↳ Holevo theorem.

Alice Bob



$X \xrightarrow{\text{classical}} Y$

msg. 02001001

= "I'm hungry"

$p = 0.0001$

Space of Msgs

I'm hungry ~ .

:-) It's raining in HK

Holevo Thm:

Von Neumann Entropy

$$\underbrace{I(X;Y)}_{} \leq \overbrace{S(\rho)} - \sum_i p_i S(\rho_i).$$

mutual Info: $I(X;Y) := \underbrace{H(X)}_{\text{Shannon's entropy}} - H(X|Y)$

$$H(X) = \sum_i p_i \log\left(\frac{1}{p_i}\right)$$

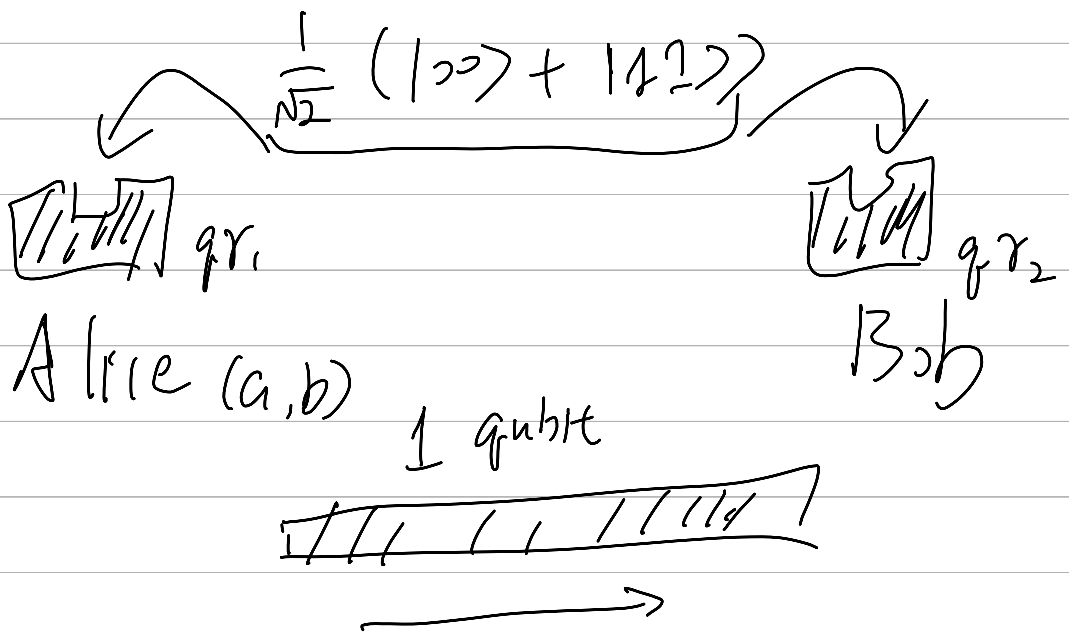
$X = x_i$

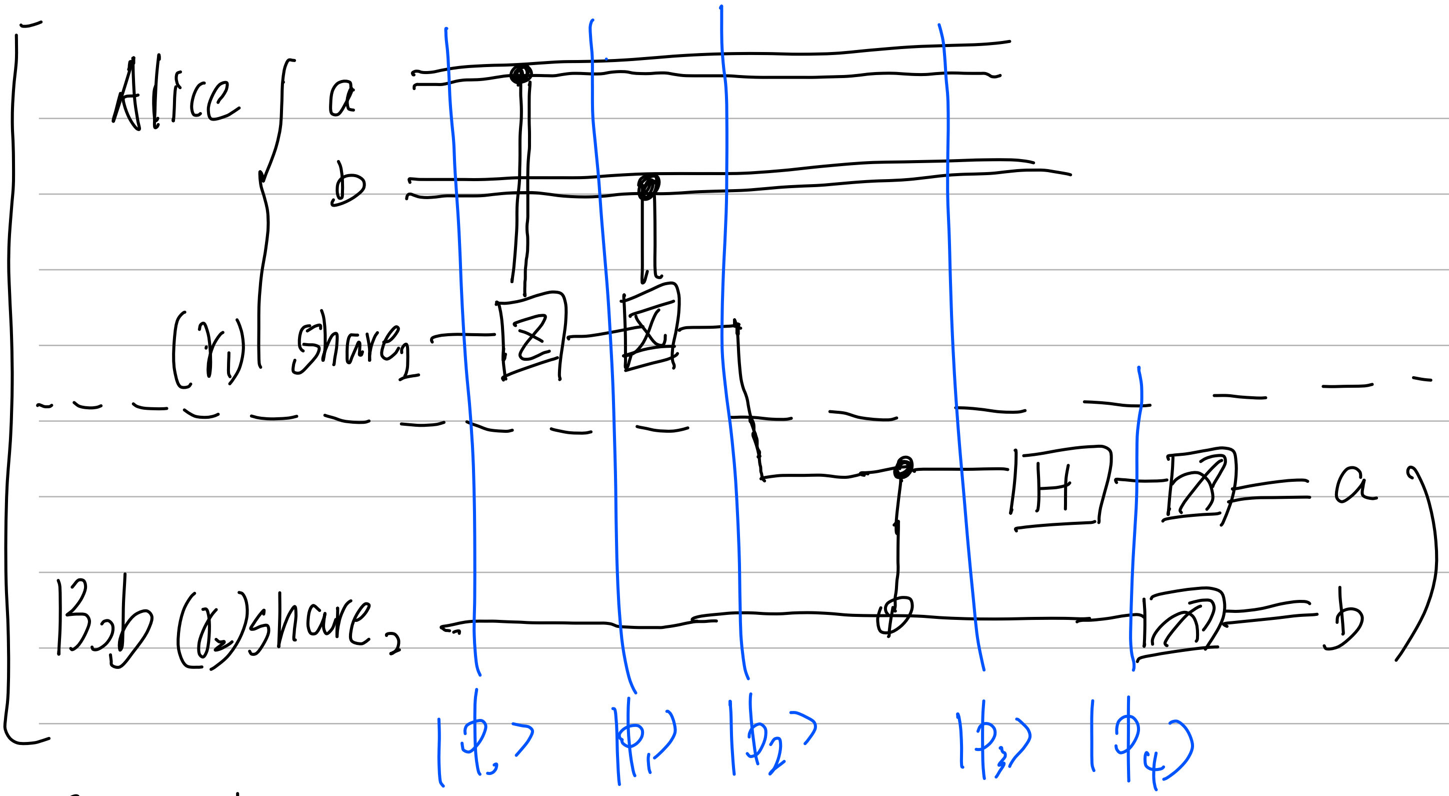
Corollary: (of Holevo Thm)

An n-qubit channel can reliably transmit at most n classical bits

High-level Idea:

With the help of "Entanglement", we can communicate n bits with $< n$ qubits. (thus, bypassing the lower bound established by Holevo's thm)





Protocol:

Alice:

a: If $a=1$, apply Pauli $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ to share$_1$

If $a=0$, do nothing.

b: If $b=1$, apply Pauli $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

If $b=0$, do nothing.

- Alice $\xrightarrow{\text{share}_2}$ Bob

Bob: ① Apply CNOT to $|\cdots\rangle_{r_1 r_2}$

② Apply H to $|\cdots\rangle_{r_1}$

③ Measure $|\cdots\rangle_{r_1 r_2}$ in standard basis

$\hookrightarrow (a, b)$

$(a=1\ \ b=0)$ - Case:

$$|\phi_0\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle_{r_1 r_2} + |11\rangle_{r_1 r_2}\right)$$

$$|\phi_1\rangle = \left(Z^a_{r_1} \otimes \mathbb{I}_{r_2}\right) \cdot |\phi_0\rangle$$

$$= \frac{1}{\sqrt{2}}\left(|00\rangle_{r_1 r_2} - |11\rangle_{r_1 r_2}\right) \quad (\text{when } a=1)$$

$$|\phi_2\rangle = \left(X^b_{r_1} \otimes \mathbb{I}_{r_2}\right)|\phi_1\rangle$$

$$= |\phi_1\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle - |11\rangle\right) \quad (\text{when } b=0)$$

$$|\phi_3\rangle = CNOT_{r_1, r_2}|\phi_2\rangle$$

$$= \frac{1}{\sqrt{2}}\left(|00\rangle_{r_1 r_2} - |10\rangle_{r_1 r_2}\right)$$

$$\boxed{\begin{cases} H|+\rangle = |0\rangle \\ H|-\rangle = |1\rangle \end{cases}}$$

$$|\phi_4\rangle = \left(H_{r_1} \otimes \mathbb{I}_{r_2}\right) \underbrace{\frac{1}{\sqrt{2}}\left(|0\rangle_{r_1} - |1\rangle_{r_1}\right)}_{|-\rangle} |0\rangle_{r_2}$$

$$= \underline{|1\rangle_{r_1}|0\rangle_{r_2}}$$

by

$\downarrow$ Measure in $(|0\rangle, |1\rangle)$ basis

w.p. 1, Bob learns $(1, 0)$

$$= (a, b)$$
$$a = 1, \quad b = 0$$

Summary:

Super Dense coding:

$$\underbrace{1 \text{ ebit}}_{\substack{\text{entangled qubit} \\ = \\ \text{EPR pair}}} + 1 \text{ qubit} = 2 \underbrace{\text{cbits}}_{\text{classical bits}}$$

Teleportation:

$$1 \text{ ebit} + 2 \text{ cbits} = 1 \text{ qubit}$$

EPR Paradox and CHSH Game:

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

EPR: Einstein, Podolsky, Rosen

Alice — Bob

Alice: qr_1 = share$_1$

Bob: qr_2 = share$_2$

$|00\rangle_{r_1 r_2}$

w.p. $\begin{cases} \frac{1}{2} & \text{observe } |0\rangle_{r_1} \text{ ——— } |0\rangle_{r_2} \\ \frac{1}{2} & \text{observe } |1\rangle \text{ ——— } |1\rangle_{r_2} \end{cases}$

$|1\rangle_{r_1} |1\rangle_{r_2}$

$c = 3 \times 10^8$ m/s

Question 1:

- Infor. travels faster than light ???

  No. Propagated physical effects by EPR pair cannot be used to carry info. reliably.

EPR's thoughts:

- Realism: Measurements simply reveals the intrinsic properties of a system
  observation

  It cannot create properties!!!

- Locality: Physical effects/influences can't propagate faster than ~~light~~ the speed of light.

Why do EPR believe that?

Realism: ① compatible with Classical Phy.

② compatible with every day experience.

Locality: ① Special Relativity.

② Principle of local Causality.

"Spooky action at distance"

QM isn't a complete theory.

- Hidden variable.

$\alpha|0\rangle + \beta|1\rangle$