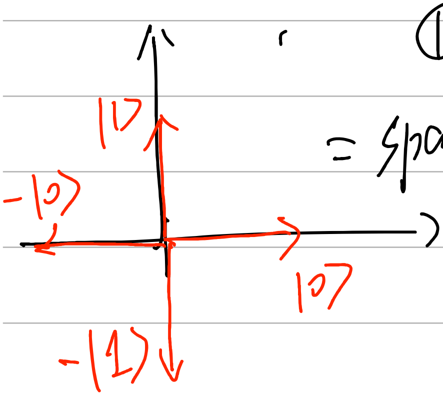
Recap: Postulate 3 (Born's Rule / Measurement)

- $\{M_m\}_{m\in I}$, I : index set. (labels for M-outcome)
- (Completeness) $\sum_{m\in I} M_m^\dagger M_m = \mathbb{I}$
- M. outcome : m
  - Prob : $p(m) = \langle\psi| M_m^\dagger M_m |\psi\rangle$
  - Post-M state: $\dfrac{M_m \cdot |\psi\rangle}{\| M_m \cdot |\psi\rangle \|}$ → Inner-product induced norm

$$\| M_m \cdot |\psi\rangle \| = \sqrt{\langle\psi| M_m^\dagger M_m |\psi\rangle}$$

Phase:



$\mathbb{C}^2$

$= \text{span}\{|0\rangle, |1\rangle\}$ set of orthonormal basis

$-|0\rangle, |1\rangle$

$-|0\rangle, -|1\rangle$

$\vec{u}_1, \vec{u}_2 \cdots \vec{u}_n$

$\|$

$|u_1\rangle \quad |u_2\rangle \cdots \quad |u_n\rangle$

$\| \quad \| \quad \cdots \quad \|$

$|1\rangle \quad |2\rangle \quad |n\rangle$

$$-|0\rangle = -\begin{bmatrix}1\\0\end{bmatrix}$$

$$\| -|0\rangle \| = |-1| \cdot \| |0\rangle \|$$

If $\{|0\rangle, -|0\rangle\}$ are different / same.

- Design a measurement to distinguish them. ? $\{2, \sqrt{4}\}$

can you ?

- No. you can't. (Provably)

Thm: $|\psi\rangle$ vs. $\{\underbrace{|\phi\rangle = e^{i\theta}|\psi\rangle}_{\text{state}}\}_{\theta \in [0, 2\pi)}$

can't be distinguished by any measurement

$\| |\phi\rangle \| = \underbrace{|e^{i\theta}|}_{=1} \cdot \| |\psi\rangle \|$

$|e^{i\theta}| = \sqrt{(e^{i\theta})^* \cdot e^{i\theta}} = \sqrt{e^{-i\theta} \cdot e^{i\theta}} = 1$

Proof: consider any $\{M_m\}$.

$|\psi\rangle$: w.p. $p(m) = \langle\psi| M_m^\dagger M_m |\psi\rangle$, observe m, Post-M: $\dfrac{M_m|\psi\rangle}{\sqrt{p(m)}}$

$e^{i\theta} = \cos\theta + i\sin\theta$

$|e^{i\theta}| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$

$|\phi\rangle$: w.p. $p(m) = \langle\phi| M_m^\dagger M_m |\phi\rangle$, observe m: Post-M: $\dfrac{e^{i\theta} M_m|\psi\rangle}{\sqrt{p(m)}}$

$$\downarrow \quad |\phi\rangle = (e^{i\theta}|\psi\rangle)$$

$$\langle\psi| e^{-i\theta} \cdot M_m^\dagger M_m e^{i\theta} |\psi\rangle$$

$$= e^{-i\theta} \cdot e^{i\theta} \langle\psi| M_m^\dagger M_m |\psi\rangle$$

In POVM: post-M "nuclear waste"

(Global phase doesn't matter)

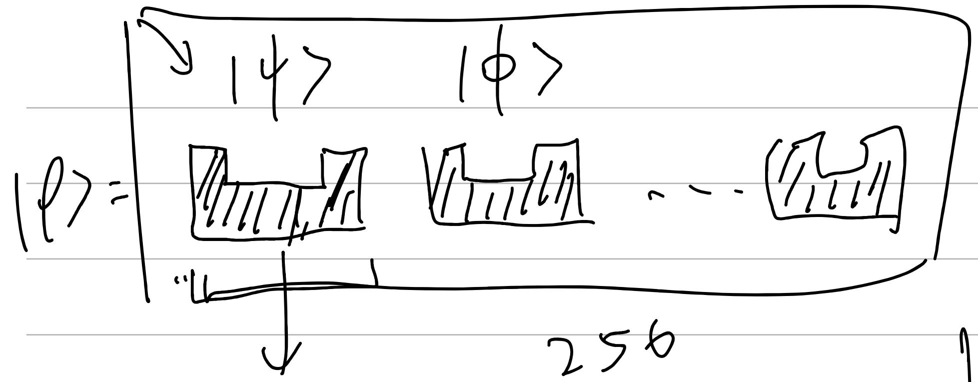
Relative phase matters:

$$|0\rangle \underbrace{+}_{\text{Relative phase}} |1\rangle \overset{\text{diff.}}{\sim} |0\rangle \underbrace{-}_{\text{Relative phase}} |1\rangle$$

$$\begin{cases} |0\rangle + |1\rangle = -(|0\rangle + |1\rangle) = -|0\rangle - |1\rangle \\ |0\rangle - |1\rangle = -(|0\rangle - |1\rangle) = |1\rangle - |0\rangle \end{cases}$$

Postulate 4: Composed Q-system.

$|\psi\rangle = \mathbb{C}^n \qquad |\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle + \cdots + \alpha_{n-1}|n-1\rangle$



256 qbits

2^{256} - dimensional vector space

$\text{span}\{|0\rangle, |1\rangle\} = \mathbb{C}^2$

$|\varphi\rangle \in \mathbb{C}^{2^{256}}$

John Watrous: CS766/QIC820
Theory of Quantum Information
(Fall 2011) lecture notes.

Rigorous treatment to postulate 4.

Kronecker Product (for Matrices):

def: let A be $m \times n$ matrix over $\mathbb{C}$
B be $p \times q$ ..

The Kronecker product between A and B is defined as:

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & \cdots & b_{1q} \\ \vdots & & \\ b_{p1} & \cdots & b_{pq} \end{bmatrix}$$

$$\underbrace{A \otimes B}_{\text{reads "tensor}} = \underbrace{\begin{bmatrix} (a_{11} \cdot B)_{p\times q} & (a_{12} \cdot B)_{p\times q} & \cdots & a_{1n} \cdot B \\ \vdots & & & \\ a_{n1} \cdot B & \cdots & & a_{mn} \cdot B \end{bmatrix}}_{(m \cdot p \times n \cdot q)\text{ - matrix.}}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \otimes \begin{bmatrix} 0 & 5 \\ 6 & 7 \end{bmatrix} = \begin{bmatrix} 1 \cdot \begin{bmatrix} 0 & 5 \\ 6 & 7 \end{bmatrix} & 2 \cdot \begin{bmatrix} 0 & 5 \\ 6 & 7 \end{bmatrix} \\ 3 \cdot \begin{bmatrix} 0 & 5 \\ 6 & 7 \end{bmatrix} & 4 \cdot \begin{bmatrix} 0 & 5 \\ 6 & 7 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 5 & 0 & 10 \\ 6 & 7 & 12 & 14 \\ 0 & 15 & 0 & 20 \\ 18 & 21 & 24 & 28 \end{bmatrix}$$

Properties of Kronecker product:

1: (Mixed - Product Property)

Let A, B, C, D. matrices,

$$(A \otimes B) \cdot (C \otimes D) = (A \cdot C) \otimes (B \cdot D)$$

[as long as their dimensions allows you to compute AC and BD]

2. $(A \otimes B)^{\dagger} = A^{\dagger} \otimes B^{\dagger}$ $\quad [(A \cdot B)^{\dagger} = B^{\dagger} \cdot A^{\dagger}]$

$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$ (if A and B are invertable)

3. Non-commutativity: In general, $A \otimes B \neq B \otimes A$

4. $tr(A \otimes B) = tr(A) \cdot tr(B)$

$[tr(AB) \neq tr(A) \cdot tr(B)]$

5. ("As-you-expected" Properties):

$$\begin{cases} A \otimes (B + C) = A \otimes B + A \otimes C \\ (B + C) \otimes A = B \otimes A + C \otimes A \end{cases}$$

$(k \cdot A) \otimes B = A \otimes (k \cdot B) = k \cdot (A \otimes B)$ $\quad k \in \mathbb{C}$.

$(A \otimes B) \otimes C = A \otimes (B \otimes C)$

$A \otimes 0 = 0 \otimes A = 0$ $\quad$ (0 is the 0-matrix)

Hilbert Space formed via Kronecker Product:

Give $\mathbb{C}^n$ and $\mathbb{C}^m$. Define a set.

$$\boxed{\mathbb{C}^n \otimes \mathbb{C}^m} := \text{span}\{\vec{v} \underbrace{\otimes}_{\text{is the Kronecker product.}} \vec{w} \mid \vec{v} \in \mathbb{C}^n, \vec{w} \in \mathbb{C}^m\}$$

just a symbol, meaningless so far.

is the Kronecker product.

Switch to Dirac notation.

$$\mathbb{C}^n \otimes \mathbb{C}^m := \text{span}\{|v\rangle \otimes |w\rangle \mid |v\rangle \in \mathbb{C}^n, |w\rangle \in \mathbb{C}^m\}$$

Theorem 1: $\forall\ \mathbb{C}^n, \mathbb{C}^m$, the set $\mathbb{C}^n \otimes \mathbb{C}^m$ is a Hilbert space (over $\mathbb{C}$) under:

(1) (Vector addition): $\underbrace{|v_1\rangle \otimes |w_1\rangle}_{n\cdot m \times 1} + \underbrace{|v_2\rangle \otimes |w_2\rangle}_{n\cdot m \times 1}$ (element-wise addition)

(2) (scalar Multiplication):

$$a \cdot \underbrace{(|v\rangle \otimes |w\rangle)}_{n\cdot m \times 1} = (a\cdot|v\rangle) \otimes |w\rangle$$

$$= |v\rangle \otimes (a\cdot|w\rangle)$$

element-wise

③ (Inner product):

$$\text{Inner}(|v_1\rangle \otimes |v_2\rangle,\ |w_1\rangle \otimes |w_2\rangle) := (|v_1\rangle \otimes |v_2\rangle)^{\dagger} \cdot (|w_1\rangle \otimes |w_2\rangle)$$

(by the properties of Kronecker product)

$$(\langle v_1| \otimes \langle v_2|)(|w_1\rangle \otimes |w_2\rangle)$$

$$= \underline{\langle v_1|w_1\rangle} \otimes \underline{\langle v_2|w_2\rangle}$$

$$= \langle v_1|w_1\rangle \langle v_2|w_2\rangle$$

Notational Remark:

$$|v\rangle \otimes |w\rangle = |v\rangle|w\rangle = |v, w\rangle = |vw\rangle$$

$\mathbb{C}^n \otimes \mathbb{C}^m$: called "tensor product" space of $\mathbb{C}^n, \mathbb{C}^m$ $\neq |\phi\rangle$

Postulate 4 (Composed Q-system)

- The state space of a composite Q-system is the tensor product of the state space of its component Q-systems.

Some Examples:

- $|0\rangle \otimes |1\rangle = |01\rangle$

$$\underbrace{\underset{\mathbb{C}^2}{\cap} \quad \underset{\mathbb{C}^2}{\cap}}_{\mathbb{C}^4}$$

$$(A+B)\otimes C = A\otimes C + B\otimes C$$

- $(\alpha_0|0\rangle + \alpha_1|1\rangle) \otimes (\beta_0|0\rangle + \beta_1|1\rangle)$

$= \alpha_0|0\rangle \otimes (\beta_0|0\rangle + \beta_1|1\rangle) + \alpha_1|1\rangle(\sim)$

$$= \underbrace{\alpha_0\cdot\beta_0\cdot|00\rangle + \alpha_0\beta_1|0,1\rangle + \alpha_1\cdot\beta_0\cdot|10\rangle + \alpha_1\beta_1|11\rangle}$$

- $\frac{1}{\sqrt{2}}\underbrace{(|00\rangle + |11\rangle)}$ - EPR pair.

can't be expressed as $|\alpha\rangle \otimes |\beta\rangle$

assume for contradiction.

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \stackrel{?}{=} (\alpha_0|0\rangle + \alpha_1|1\rangle)\otimes(\beta_0|0\rangle + \beta_1|1\rangle)$$

$$\Rightarrow \left\{ \begin{array}{l} \alpha_0\beta_0 = \frac{1}{\sqrt{2}} \\ \alpha_1\beta_0 = \alpha_0\beta_1 = 0 \\ \alpha_1\cdot\beta_1 = \frac{1}{\sqrt{2}} \end{array} \right)$$

Linear Operators for tensor product space.

$$\text{game.}\left[\begin{array}{ccc} |v\rangle & \otimes & |w\rangle \\ \updownarrow & & \updownarrow \\ U_1 & & U_2 \end{array}\right.$$

$$(A\otimes B)\cdot(C\otimes D) = (AC\otimes BD)$$

$$= (U_1 \otimes U_2)\cdot(|v\rangle \otimes |w\rangle)$$

$$= (U_1|v\rangle) \otimes (U_2|w\rangle)$$

$$\boxed{\text{Prove}: U_1 \otimes U_2 \text{ is a unitary if } U_1, U_2 \text{ are}}$$

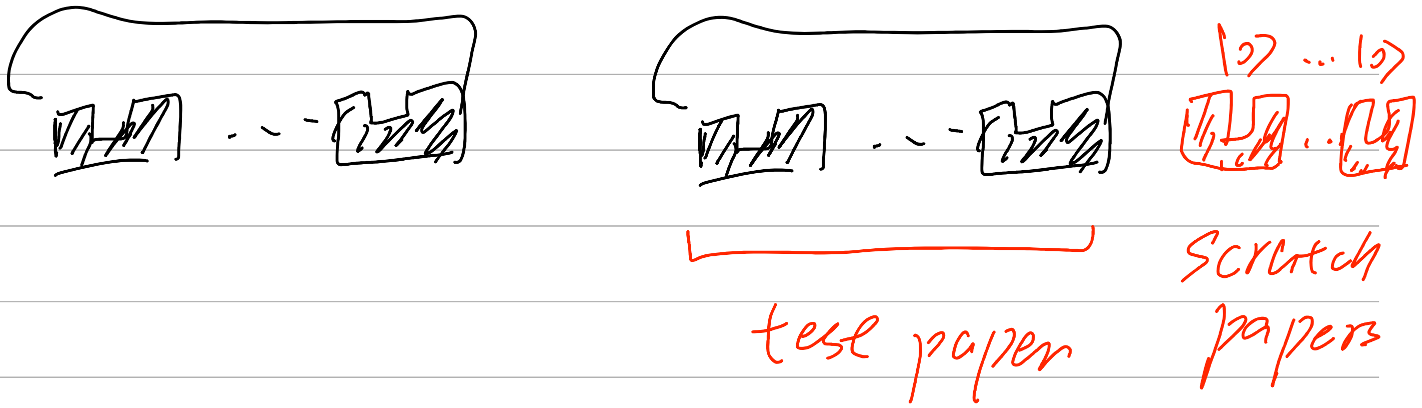
$$|00\rangle \xrightarrow{U} \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$U := \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}$$

$$|00\rangle = |0\rangle \otimes |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

General $\{M_m\}_m$ $\Longleftrightarrow$ Projective $\{M'_m\}_m$

└ up to ancilla qubits,



Nielsen-Chuang, Sec. 2.2.8 Composite systems

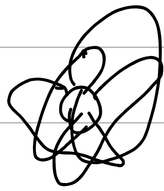
toward the end

Basic Quantum-Exclusive Effects

1. No-cloning
2. quantum teleportation
3. superdense encoding
4. EPR paradox (CHSH game)

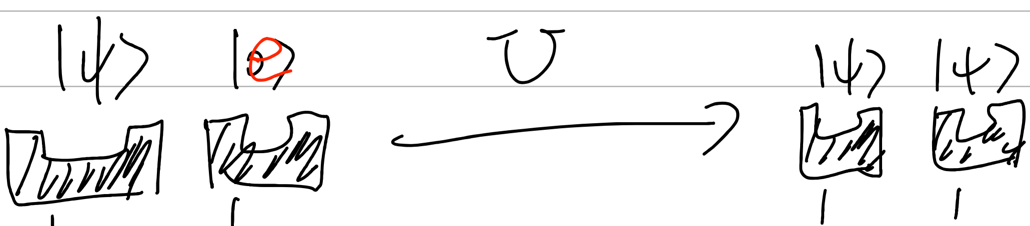
No-cloning

$\alpha|0\rangle+\beta|1\rangle$



$\sqrt{0.3}\,|0\rangle+\sqrt{0.7}\,|1\rangle$

Thm: $\forall$ Hilbert space $\mathcal{H}$, there is no unitary U on $\mathcal{H}\otimes\mathcal{H}$ such that for all state $|\psi\rangle_A \in \mathcal{H}$ and $|e\rangle_B \in \mathcal{H}$: $U(|\psi\rangle_A|e\rangle_B) = |\psi\rangle_A|\psi\rangle_B$.



$\psi_A \quad \psi_B \qquad\qquad \psi_A \quad \psi_B$

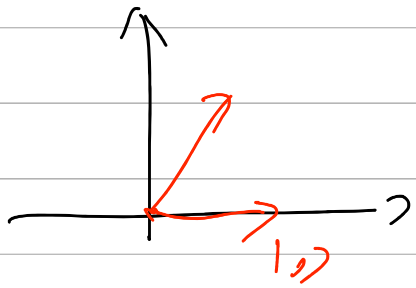
$\forall\, |\psi\rangle, \ \exists\, U_{|\psi\rangle}$ s.t.

$$U_{|\psi\rangle}\, |\psi\rangle|0\rangle \longrightarrow |\psi\rangle|\psi\rangle$$

If $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ $\quad |\alpha|^2 + |\beta|^2 = 1$

then define $U_{|\psi\rangle}$ as:

$$U_{|\psi\rangle}\, |0\rangle \longrightarrow \alpha|0\rangle + \beta|1\rangle$$



Proof of No-Cloning:

Assume for contradiction,

$\exists\, U, \underline{|e\rangle_B}$ s.t. $\forall\, |\psi\rangle_A$

$$U|\psi\rangle_A|e\rangle_B = |\psi\rangle_A|\psi\rangle_B \quad \cdots \quad ②$$

- $|\psi\rangle_A = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

Perspective 1:

$$U\,|\psi\rangle_A|e\rangle_B = |\psi\rangle_A|\psi\rangle_A = \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)\cdot\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)$$

Perspective 2:

$$U(|\psi\rangle_A|e\rangle_B) = U\left(\left(\frac{1}{\sqrt{2}}|0\rangle+\frac{1}{\sqrt{2}}|1\rangle\right)\otimes|e\rangle_B\right)$$

$$= U\left(\frac{1}{\sqrt{2}}|0\rangle_A|e\rangle_B + \frac{1}{\sqrt{2}}|1\rangle_A|e\rangle_B\right)$$

$$= \frac{1}{\sqrt{2}}U|0\rangle_A|e\rangle_B + \frac{1}{\sqrt{2}}U|1\rangle_A|e\rangle_B$$

$$= \frac{1}{\sqrt{2}}|0\rangle_A|0\rangle_B + \frac{1}{\sqrt{2}}|1\rangle_A|1\rangle_B$$

$$+\,0\,|0\rangle_A|1\rangle_B + 0\cdot|1\rangle_A|0\rangle_B$$