For remaining 3 postulates. (roughly follow Nielsen-Chuang)

Postulate 2 : (State Evolution)

The evolution of a closed quantum system is described by a unitary operator.

Notation-wise :

$$\underset{t=0}{|\psi(0)\rangle} \xrightarrow{\Delta t} |\psi(0+\Delta t)\rangle$$

$$|\psi(\Delta t)\rangle = \underbrace{U_{\Delta t}}_{\text{unitary}} |\psi(0)\rangle$$

$U^{\dagger} = U^{-1}$

$\left( \Leftrightarrow U^{\dagger} \cdot U = I \right.$
$\left. \Leftrightarrow U \cdot U^{\dagger} = I \right)$

Heisenberg's picture.

↓

Shrödinger's

$H^{\dagger} = H$ Hermitian

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = H |\psi(t)\rangle$$

$\rightarrow \langle H\cdot\vec{x}, \vec{y}\rangle = \langle \vec{x}, \tilde{H}\vec{y}\rangle$

$\tilde{H}$ is the adjoint of $H$

$\tilde{H} = H^{\dagger}$

First-order

Ordinary Differential Equation (ODE)

$$|\psi(t)) = \cos(f(t)) \quad \text{(calculus)}$$

$$\downarrow \frac{d}{dt}$$

$$= -\sin(f(t))\cdot f'(t)$$

Complex Field:

$$|\psi(t)\rangle = e^{if(t)} = \cos(f(t)) + i\sin(f(t))$$

Solution to S. eq:

$$\boxed{|\psi(\Delta t)\rangle = e^{iH\Delta t}\cdot|\psi(0)\rangle}$$

need to define functions on Matrices / (normal) linear operators.

$\Leftrightarrow$ spectral decomp.

$f(x)$ : $x \in \mathbb{C}$

$\Rightarrow$ $M$ is normal:

$$\Leftrightarrow M = \sum_j \lambda_j |\phi_j\rangle\langle\phi_j| \quad \text{(Spectral decomp)}$$

$$f(M) := \sum_j f(a_j) |\phi_j\rangle\langle\phi_j|$$

where $\{|\phi_j\rangle\}_j$ is a set of orthonormal basis of $V$.
and they happen to be a complete set of M's eigenvectors.

solution of S. eq $\Rightarrow$ Postulate 2.
↳ $e^{iHt}$ is unitary.

- Simplified QM (over Real numbers)

orthonormal matrices:

$\mathbb{R} \ni Q^T \cdot Q = Q \cdot Q^T = I$

$\mathbb{C} \ni U^\dagger \cdot U = U \cdot U^\dagger = I$

$\Rightarrow$ orthonormal matrices are unitaries.

✓ $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

✓ $H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ $\begin{cases} H|0\rangle = |+\rangle \\ H|1\rangle = |-\rangle \end{cases}$

Pauli's matrices:

$$\underline{X} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \underline{Y} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad \underline{Z} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$\mathbb{I}$ $\quad (i, -i)$ $\qquad\qquad |+\rangle, |-\rangle$

---

Postulate 3: Measurement (Born's Rule)

$\alpha|0\rangle + \beta|1\rangle$ $\quad \alpha|\uparrow\rangle + \beta|\downarrow\rangle$ $\quad$ Max Born

$\{\uparrow, \downarrow\}$ $\quad$ advisor of Heisenberg.

A measurement is defined by a set of matrices $\{M_m\}_{m \in \text{Index}}$ s.t.

↳ by your choice

- (completeness) $\displaystyle\sum_m M_m^\dagger M_m = \mathbb{I}$ ☆

Measure $|\psi\rangle$ using $\{M_m\}$, you get.

- w.p. $p(m) = \langle\psi|M_m^\dagger (M_m|\psi\rangle)$

you get $\dfrac{M_m|\psi\rangle}{\sqrt{\langle\psi|M_m^\dagger M_m|\psi\rangle}}$ as the post-M state.

$$= \frac{M_m|\psi\rangle}{\| M_m|\psi\rangle \|}$$

you observe m as the M-outcome.

$$|\psi\rangle \xrightarrow[\Delta t]{} \underbrace{U|\psi\rangle}$$

$\underline{M_m|\psi\rangle}$ ~ not qualified to be a Q-state

normalize

$$\left\| \frac{\vec{v}}{\|\vec{v}\|} \right\| = 1$$

$$\frac{M_m|\psi\rangle}{\underline{\| M_m|\psi\rangle \|}}$$ ~ unit vector.

$$\boxed{\text{Inner product induced Norm: } \quad \|\vec{v}\| := \sqrt{\langle \vec{v}, \vec{v} \rangle}}$$

$\hookrightarrow$ $\| M_m|\psi\rangle \| := \sqrt{\langle M_m|\psi\rangle, M_m|\psi\rangle \rangle}$

$$= \sqrt{(M_m|\psi\rangle)^\dagger \cdot (M_m|\psi\rangle)}$$

$$= \sqrt{(|\psi\rangle)^\dagger \cdot M_m^\dagger \cdot M_m|\psi\rangle}$$

$$= \sqrt{\langle\psi| \cdot M_m^\dagger \cdot M_m \cdot |\psi\rangle}$$

$|\psi\rangle$

$$\sum_m p(m) \overset{?}{=} 1$$

$$\Leftarrow \quad \sum_m \langle\psi| M_m^\dagger \cdot M_m |\psi\rangle = \langle\psi| \left(\sum_m M_m^\dagger M_m\right) |\psi\rangle$$

$\sum_m M_m^\dagger M_m \to \mathbb{I}$

$$\text{(completeness)} = \langle\psi| \mathbb{I} \cdot |\psi\rangle$$

$$= \langle\psi|\psi\rangle$$

$$= \| |\psi\rangle \|^2$$

$$= 1^2 = 1$$

Positive definiteness: of any inner product:

$\langle \vec{x}, \vec{x} \rangle \geq 0$ with equality holds at $\vec{x} = \vec{0}$

First example:

Measure $|\psi\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle$ in stand. basis

$\{M_\uparrow, M_\downarrow\}$

Hermitian: $M_\uparrow^\dagger = M_\uparrow$

Projector: $M_\uparrow^2 = M_\uparrow$

$M_\uparrow := |\uparrow\rangle\langle\uparrow|$

$M_\downarrow := \mathbb{I} - M_\uparrow^\dagger M_\uparrow = |\downarrow\rangle\langle\downarrow|$

$= \mathbb{I} - M_\uparrow : M_\uparrow$

$= \mathbb{I} - M_\uparrow$

$\|$

For a set of orthonormal basis $\{|\phi_j\rangle\}$:
Then:

$$\sum_j |\phi_j\rangle\langle\phi_j| = \mathbb{I}$$

$\Downarrow$

$= |\downarrow\rangle\langle\downarrow|$

$$M_\uparrow^\dagger = (|\uparrow\rangle\langle\uparrow|)^\dagger$$
$$= (\langle\uparrow|)^\dagger (|\uparrow\rangle)^\dagger$$
$$= |\uparrow\rangle\langle\uparrow| = M_\uparrow$$

$$M_\uparrow^2 = (|\uparrow\rangle\langle\uparrow|)(|\uparrow\rangle\langle\uparrow|)$$
$$= |\uparrow\rangle \underbrace{(\langle\uparrow|\uparrow\rangle)}_{1} \langle\uparrow|$$
$$= |\uparrow\rangle\langle\uparrow|$$

$$\boxed{M = \sum_j \lambda_j |\phi_j\rangle\langle\phi_j|}$$

$|\psi\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle$

w.p. $p(\uparrow) = \langle\psi| M_\uparrow^\dagger M_\uparrow |\psi\rangle$

$$= \langle\psi| M_\uparrow |\psi\rangle$$

$$= \underbrace{\left[(\alpha|\uparrow\rangle)^\dagger + (\beta|\downarrow\rangle)^\dagger\right]}_{\langle\psi|} |\uparrow\rangle\langle\uparrow| \overbrace{(\cdots)}^{|\psi\rangle}$$

$$\boxed{(\alpha \cdot |\psi\rangle)^\dagger \neq \alpha \cdot \langle\psi|}$$

$|\psi\rangle$

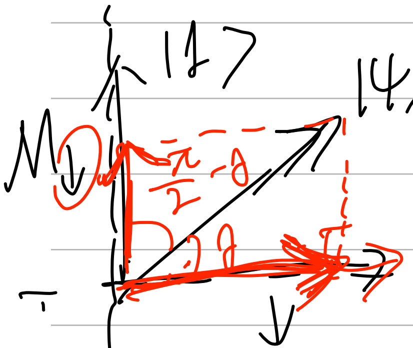
$$\boxed{= \alpha^* \cdot \langle\psi|}$$

$$= (\alpha^* \langle\uparrow| + \beta^* \langle\downarrow|)\, |\uparrow\rangle\langle\uparrow|\, (\ldots)$$

$$= (\alpha^* \cdot 1 + 0)(\alpha \cdot 1 + 0)$$

$$= \alpha^* \cdot \alpha = |\alpha|^2$$

$$\frac{M_\uparrow \cdot |\psi\rangle}{\sqrt{\langle \ldots \rangle}} = \frac{|\uparrow\rangle\langle\uparrow|(\alpha|\uparrow\rangle + \beta|\downarrow\rangle)}{\sqrt{|\alpha|^2}}$$

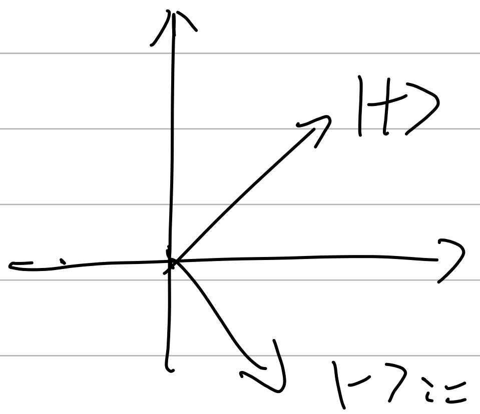



$$= \frac{\alpha|\uparrow\rangle}{|\alpha|} = \underbrace{\frac{\alpha}{|\alpha|}}_{\text{"global phase"}} \cdot |\uparrow\rangle$$

$\mathbb{C}$

$M_\uparrow = |\uparrow\rangle\langle\uparrow|$

(physically $\equiv |\uparrow\rangle$)



$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) := H|0\rangle$

$|-\rangle := \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) := H|1\rangle$

$\underbrace{\{M_+, M_-\}}$

$M_+ := |+\rangle\langle+|$

$M_- := |-\rangle\langle-|$

↳ Measurement under Hadamard bais.

---

In general, measurement matrices may not be projectors.

Special Types of Measurement:

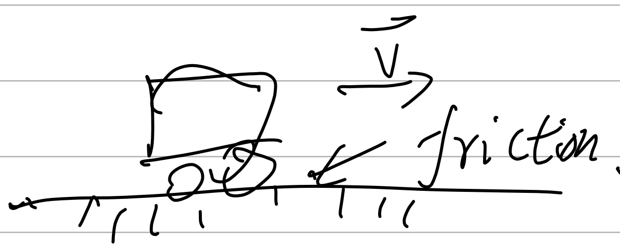
① Projective M...

② Positive Operator-Valued M... (POVM)

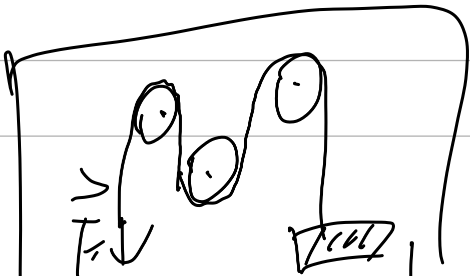
Projective M: - -

---

- Projective M... are M... where $\{M_m\}_m$ are projectors.




$$\boxed{\vec{F} = \frac{d\vec{p}}{dt}}$$




Lagrange - Euler

- Hamilton

$\downarrow mg$

H : a Hamiltonia of a system

$\downarrow$ matrix.

In quantum setting:

H happens to be a Hermitian matrix

$$H \overset{\text{spectral}}{=\!=\!=} \sum_j \underbrace{\lambda_j}_{} \underbrace{|\phi_j\rangle\langle\phi_j|}_{\text{eigen vectors of } H}$$

must be real for Hermitian H

Set $M_j = |\phi_j\rangle\langle\phi_j|$

- $\{M_j\}_j$ is a set of projetors.
- $$\sum_j |\phi_j\rangle\langle\phi_j| = \mathbb{I} = \sum_j M_j \overset{\text{Hermitian}}{\underset{\text{proj.}}{=}} \sum_j M_j^\dagger M_j$$

H is called an "observable"

Hamiltonia $\equiv$ Hermitian $\equiv$ observables

$$\overset{\uparrow}{|0\rangle\langle 0|} - \overset{\downarrow}{|1\rangle\langle 1|} = \underline{X}$$

$\underline{M_0}$ $\underline{M_1}$ Pauli

Phycists def of Projectiv M..

Let $H$ be a Hermitian, called "observables"

Spe. decomp.

$$\underline{H} = \sum_m m \underbrace{|\phi_m\rangle\langle\phi_m|}_{M_m}$$

Remark:

eigenvalues could repeat.

$m = 1$ : $\{|\phi_1^a\rangle, |\phi_1^b\rangle, |\phi_1^c\rangle\}$

$m = 2$ : $|\phi_2\rangle$

$$H = 2 \cdot |\phi_2\rangle\langle\phi_2| + 1 \cdot \left(|\phi_1^a\rangle\langle\phi_1^a| + |\phi_1^b\rangle\langle\phi_1^b| + |\phi_1^c\rangle\langle\phi_1^c|\right)$$

Most general form:

$H = \sum_m m \cdot \underbrace{P_m}_{M_m}$ , $P_m$ is a Projector and a Hermitian

$M_m^\dagger M_m = P_m^\dagger P_m = P_m$

w.p. $p(m) = \langle \psi | P_m | \psi \rangle$

observe. outcome $m$.

state collapses to $\dfrac{P_m \cdot |\psi\rangle}{\| P_m \cdot |\psi\rangle \|}$

$|\psi\rangle$

$H = \sum_m m P_m$

$$\mathbb{E}_{|\psi\rangle}(H) = \sum_m m \cdot p(m)$$

$= M_m^\dagger M_m$

$$= \sum_m m \cdot \langle \psi | P_m | \psi \rangle$$

$$= \langle \psi | \underbrace{\left( \sum_m m P_m \right)} | \psi \rangle$$

$$= \langle \psi | H \cdot | \psi \rangle$$

- Heisenberg's uncertainty Principle

  ↳ leave it later when we discuss Hamiltonian Complexity

- Projective M... are equivalent to general measurements as in Postulate 3

- $\{M_m\}$ convert? $\{P_m\}$ ↳ projector?

  ↓ ↓

  same outcome statistics.

  Yes, up to an ancilla system

  $|\psi\rangle|0\rangle$

  extend Hilbert Space

# POVM / positive operator-valued M...

- w.p. $p(m)$, see outcom m.
- don't care about post-measurement state $\frac{M_m|\psi\rangle}{\|M_m|\psi\rangle\|}$

$p(m) = \langle\psi| \underbrace{M_m^\dagger M_m} |\psi\rangle$

POVM is a set of matrices

$E_m$ s.t. $\begin{cases} \sum_m E_m = I \\ E_m \text{ is a positive operator.} \end{cases}$

$= (M_m^\dagger M_m)$

Alternative $\langle\psi| E_m |\psi\rangle \geq 0$
def of positive operators