Equip Vector Space with Geometry.

Agenda:

① Metric.
② Norm.
③ Inner Product.

$\downarrow$ roughly: more and more structures.

Metric Space*: (Don't require it to be a VS)

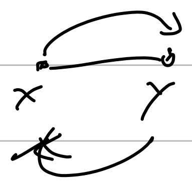
(X, d)

$X \to$ set, $d \to$ distance function.

$d: \underset{v_1}{X} \times \underset{v_2}{X} \longrightarrow \mathbb{R}$

① $d(x, x) = 0$

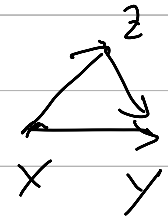
② (Positivity): if $x \neq y$, $d(x,y) > 0$

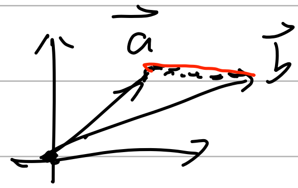


③ (Symmetry:) $d(x, y) = d(y, x)$

④ (Triangle Inequality)

$$d(x, y) \leq d(x, z) + d(z, y)$$





$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

$$d(\vec{a}, \vec{b}) = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}$$

Valid def. for dist. $\longrightarrow$ $d(\vec{a}, \vec{b}) = \sqrt{|a_1 - b_1| + |a_2 - b_2|}$

Normed Space: ← Vector Space

- V is a VS over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$
- $\|\cdot\| : V \to \mathbb{R}$ ↔ (length func.)

① Non-negativity: $\|x\| \geq 0$ ($\Leftrightarrow$)

② Positive definiteness: $\|x\| = 0$ iff $x = \vec{0}$

③ Scalability: $\|\lambda \cdot x\| = \lambda \cdot \|x\|$

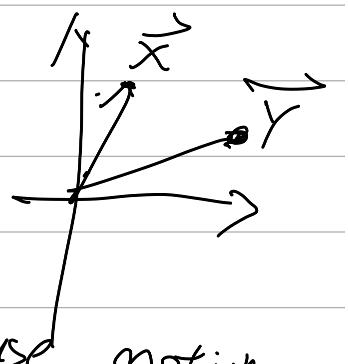
④ Triangle Inequality: $\|x+y\| \leq \|x\| + \|y\|$

Why Important?

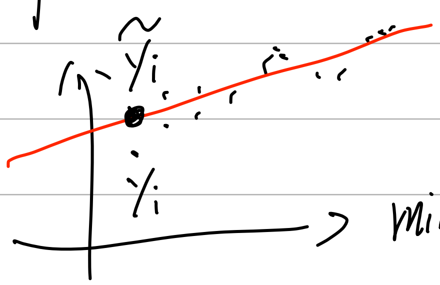
1. It allows us to talk about "length".

2. Norm-induced distance (metric):

$$d(x, y) := \|x - y\|$$



3. Optimization problems use variouse notion of Norm as their target/loss function.



Lease Squawe Method.

$$\min\left(\sum_{i=1}^{n} \|y_i - \tilde{y}_i\|\right)$$

↳ $(y_i - \tilde{y}_i)^2$

$|y_i - \tilde{y}_i|$

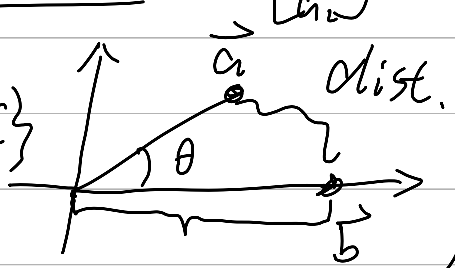
$\sqrt{3\cdot(y_i - \tilde{y}_i)^{100}}$

4. Norm-induced metric is a "natural way" to bridge between Algebraic structures and Geometric properties of a VS.

Inner-Product Space:

$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$

- V: VS over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$
- $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$



$\sqrt{a_1^2 + a_2^2}$ ↑ Euclidean

$\langle \vec{a}, \vec{b} \rangle = \|\vec{a}\| \cdot \|\vec{b}\| \cos\theta$

$\Leftrightarrow \theta := \arccos \dfrac{\langle \vec{a}, \vec{b} \rangle}{\|\vec{a}\| \, \|\vec{b}\|}$

① Linearity in the second argument:

$$\begin{cases} \langle \vec{x}, \vec{y} + \vec{z} \rangle = \langle \vec{x}, \vec{y} \rangle + \langle \vec{x}, \vec{z} \rangle \\ \langle \vec{x}, s \cdot \vec{y} \rangle = s \cdot \langle \vec{x}, \vec{y} \rangle \end{cases}$$

② $\underbrace{\text{Hermitian}}_{\text{Hermit}}$ symmetry / conjugate symmetry: $\langle \vec{x}, \vec{y} \rangle = \overline{\langle \vec{y}, \vec{x} \rangle}$

③ Positive definiteness: $\forall \vec{x} \in V$, if $\vec{x} \neq 0$, then $\langle \vec{x}, \vec{x} \rangle > 0$

Extra Requirement:

(3.5) $\forall \vec{x} \in V, \langle \vec{x}, \vec{x} \rangle \in \mathbb{R}$

$$\Rightarrow \begin{cases} \langle \vec{x}+\vec{z}, \vec{y} \rangle = \langle \vec{x}, \vec{y} \rangle + \langle \vec{z}, \vec{y} \rangle \\ \langle s\cdot\vec{x}, \vec{y} \rangle \neq s \cdot \langle \vec{x}, \vec{y} \rangle \\ \quad = \bar{s} \cdot \langle \vec{x}, \vec{y} \rangle \end{cases}$$

Why inner-product space important?

1. talk about: orthogonality, "adjoint of linear operators"
normal operator,
Hermitian operator.
"spectral decomp." $\Leftrightarrow$ eigen-decomp with orthogonal eigenvectors / basis

2. serves as a natural generalization of Euclidean Space

3. Minimal Space wher Cauchy-Schwarz inequality holds.

high school $\left(\sum_i a_i b_i\right)^2 \le \left(\sum_i a_i^2\right)\left(\sum_i b_i^2\right)$

$$\sum_i a_i b_i \le \sqrt{\sum_i a_i^2} \cdot \sqrt{\sum_i b_i^2}$$

$$\langle \vec{a}, \vec{b} \rangle \qquad \|\vec{a}\| \cdot \|\vec{b}\|$$

in Euclidean

In Inner product Space,

$$|\langle \vec{x}, \vec{y} \rangle|^2 \le \langle \vec{x}, \vec{x} \rangle \cdot \langle \vec{y}, \vec{y} \rangle$$

$$\Longleftrightarrow \quad |\langle \vec{x}, \vec{y} \rangle| \le \|\vec{x}\| \cdot \|\vec{y}\|$$

Inner product induced norm.

4. Inner product induced norm:

$$\|\vec{x}\| := \langle \vec{x}, \vec{x} \rangle$$

5: Most important space for QM.

Space $\mathbb{C}^n$; $(\mathbb{R}^n)$ ☆ (Heisenburg QM)

$$\vec{x} = (x_1, \cdots, x_n) \qquad x_i \in \mathbb{C}$$

$$\left(\langle \vec{x}, \vec{y} \rangle \xlongequal{\text{in } \mathbb{R}^n} \sum_{i=1}^n x_i \cdot y_i \right)$$

$$\langle \vec{x}, \vec{y} \rangle = \overline{\langle \vec{x}, \vec{y} \rangle}$$

$$\langle \vec{x}, \vec{y} \rangle \overset{\text{in } \mathbb{C}^n}{=\!=} \sum_{i=1}^{n} \overline{x_i} \cdot y_i$$

(shrödinger's QM)

$C[a,b]$: "continuous" ~~real-valued~~ functions over $[a,b] \subseteq \mathbb{R}$

VS: $\begin{cases} (f+g)(t) = f(t) + g(t) \\ (a \cdot f)(t) = a \cdot f(t) \end{cases}$

Inner product $\begin{cases} \langle f, g \rangle := \int_a^b f(t) \cdot g(t)\, dt & \text{(real-valued)} \\ \langle f, g \rangle := \int_a^b \overline{f(t)} \cdot g(t)\, dt & \text{(complex-valued)} \end{cases}$

Dirac Notation

$|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\langle 1| = [0 \ 1]$

$\langle 1 | 1 \rangle \sim \overbrace{\langle 1, 1 \rangle}^{\text{innerproduct.}} = [0, 1]^{\dagger} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\langle 1|$: conjugate transpose.

In space $\mathbb{C}^n$.

$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ $e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$ $e_j = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$ — j-th.

$\downarrow$ $\downarrow$ ... $\downarrow$

$|0\rangle$ $|1\rangle$ $|j-1\rangle$

orthonormal basis:

$\vec{b}_1 \in \mathbb{C}^n$ $\vec{b}_2$ ... $\vec{b}_n$

$\downarrow$ $\downarrow$ $\downarrow$

$|0\rangle$ $|1\rangle$ ... $|n-1\rangle$

$|a\rangle \xrightarrow{\text{conj. trans}} \langle a| \longrightarrow \langle a|b\rangle$ inner product.

① $|a\rangle \cdot \langle b|$

$$\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}_{n\times 1} \cdot [\bar{b}_1 \ \bar{b}_2 \cdots \bar{b}_n]_{1\times n} = \begin{bmatrix} & \end{bmatrix}_{n\times n}$$

$\in \mathbb{C}^n$

Eg) $M_{n\times n}\,|a\rangle = [M]_{n\times n}\begin{bmatrix}a_1\\ \vdots\\ a_n\end{bmatrix}$

Properties:

$$|a\rangle(\overbrace{\langle b|\cdot|c\rangle}^{\in\mathbb{C}}) = \langle b|c\rangle\cdot|a\rangle$$

$$\langle c|\,|a\rangle\langle b| = \langle c|a\rangle\cdot\langle b|$$

$$|a\rangle\langle b|\cdot M = |a\rangle(\langle b|M)$$

$$M\,|a\rangle\langle b| = (M\cdot|a\rangle)\cdot\langle b|$$

Hilbert Space.

Complete (Real or Complex) inner-product space.

$$\mathbb{Q} \quad \rightarrow \frac{1}{2!} + \frac{1}{2!} + \cdots + \cdots \frac{1}{n!} + \cdots$$

$$= e \notin \mathbb{Q}$$

$$\in \mathbb{R}$$

Postulate 1:

An isolated physical system is completely described by its state vector, which is a unit vector in a Hilbert space. $\to \mathbb{C}^n$

$\hookrightarrow$ length 1 in the inner-product induced norm

$|0\rangle \quad |2\rangle \quad \ldots \quad |n-1\rangle$ $\hookrightarrow \|\vec{x}\| = \langle \vec{x}, \vec{x} \rangle$

amplitude

$$|electron\rangle = \alpha_0 |0\rangle + \alpha_2 |1\rangle \ldots + \alpha_{n-1} \cdot |n-1\rangle$$

s.t.

$$\| electron \| = \langle electron, electron \rangle = 1$$

dagger

$$= |electron\rangle^{\dagger} \cdot |electron\rangle$$

$$= \left(\langle 0| \cdot \bar{\alpha}_0 + \langle 1| \cdot \bar{\alpha}_1 + \cdots + \langle n-1| \cdot \alpha_{n-1} \right) \cdot$$

$$= \sum_{i,j \in \{0, \cdots, n-1\}} \langle i| \cdot \bar{\alpha}_i \cdot \alpha_j |j\rangle$$

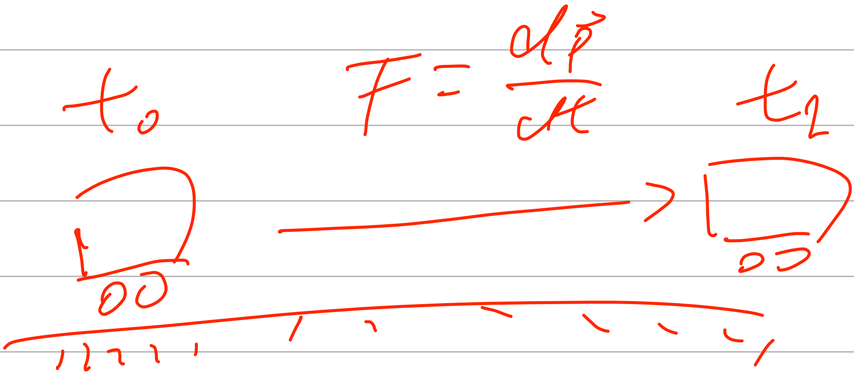
$$= \sum_{ij} \bar{\alpha}_i \alpha_j \langle i|j\rangle$$

$$= \sum_i \underbrace{\bar{\alpha}_i \alpha_i \langle i | i \rangle}_{= 1} + 0$$

$$= \sum_i \bar{\alpha}_i \alpha_i = \sum_i |\alpha_i|^2$$

$$\Rightarrow \sum_i |\alpha_i|^2 = 1$$

$$\alpha |0\rangle + \beta |1\rangle \qquad \alpha^2 + \beta^2 = 1$$



Linear Operator:

Adjoint. two inner-product space V, W, over the field $\mathbb{C}$.

For all linear operator $T : V^n \to W^m$,

there exists a unique linear operator

$\tilde{T} : W \to V$ s.t.

$$\underbrace{\langle T\vec{v}, \vec{w} \rangle}_{\text{in } W} = \underbrace{\langle \vec{v}, \tilde{T}\vec{w} \rangle}_{\text{in } V}$$

$\tilde{T}$ is called the adjoint of T.

$\tilde{T} = T^{\dagger}$ (notation

Spectral Theorem:

$A_{n \times n} = \underbrace{P}_{\text{columns of } P \text{ are linearly independent}} \Lambda P^{-1}$ [Vanilla Stretching-basis decomp.

Normal Operator / matrices:

A matrix A is called normal iff:

$$A^{\dagger} \cdot A = A \cdot A^{\dagger}$$

Spectral Theorem:

An operator A on a complex Hilbert space is **normal if and only if** it has the following eigendecomposition:

$$A = \underbrace{U}\Lambda U^\dagger$$

↳ unitary: columns are orthonormal to each other.

(Alternatively:

$$U^\dagger = U^{-1}$$
$$\iff U^\dagger \cdot U = \mathbb{I}$$
$$\iff U \cdot U^\dagger = \mathbb{I}$$

$$U = \begin{bmatrix} | & | & & | \\ \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_n \\ | & | & & | \end{bmatrix}$$

Then, the decomp $A = U\Lambda U^\dagger$ is called the spectral decomp.

$$\begin{bmatrix} | & | & & | \\ \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_n \\ | & | & & | \end{bmatrix} \qquad \Lambda = \begin{bmatrix} \lambda_0 & & 0 \\ & \lambda_1 & \\ 0 & & \lambda_{n-1} \end{bmatrix}$$

$$|0\rangle \quad |1\rangle \quad |n-1\rangle$$

$$A = (U\Lambda)U^\dagger =$$

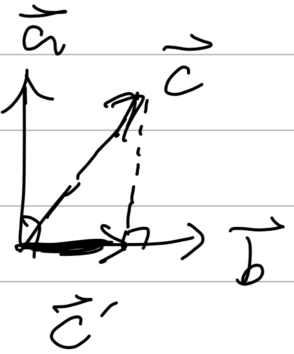
$$= [\vec{u}_1 \cdot \lambda_0 \quad \vec{u}_2 \cdot \lambda_1 \quad \cdots \quad \vec{u}_n \cdot \lambda_{n-1}] \begin{bmatrix} — & \vec{u}_1^\dagger & — \\ — & \vec{u}_2^\dagger & — \\ & \vdots & \\ — & \vec{u}_n^\dagger & — \end{bmatrix}$$

$$= \underbrace{\vec{u}_1 \cdot \vec{u}_1^\dagger} \cdot \lambda_0 + \vec{u}_2 \cdot \vec{u}_2^\dagger \cdot \lambda_1 + \ldots + \vec{u}_n \cdot \vec{u}_n^\dagger \cdot \lambda_{n-1}$$

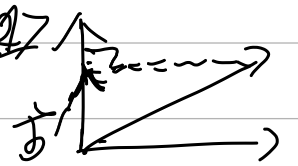
$$= |0\rangle\langle 0| \cdot \lambda_0 + |1\rangle\langle 1| \cdot \lambda_1 + \ldots + |n-1\rangle\langle n-1| \cdot \lambda_{n-1}$$

$$= \sum_{j=0}^{n-1} \lambda_j \cdot |j\rangle\langle j|$$

$|1\rangle\langle 1|$



$\begin{bmatrix} 0 \\ 1 \end{bmatrix} := |1\rangle$



$|0\rangle := \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$|b\rangle = \alpha|0\rangle + \beta|1\rangle$$

$$|1\rangle\langle 1| \cdot |b\rangle = |1\rangle\langle 1| (\alpha|0\rangle + \beta|1\rangle)$$

$$= \beta \cdot |1\rangle\underbrace{\langle 1| \cdot |1\rangle}_{1}$$

$$= \beta \cdot |1\rangle$$

$$A = \lambda_0 |0\rangle\langle 0| + \lambda_1 |1\rangle\langle 1| + \lambda_2 |2\rangle\langle 2|$$

$$\lambda_1 = \lambda_2 = \lambda$$

$$= \lambda_0 \cdot |0\rangle\langle 0| + \lambda (|1\rangle\langle 1| + |2\rangle\langle 2|)$$

$(H^\dagger \cdot H = H \cdot H^\dagger)$ — normal

Hermitian: $H^{\dagger}=H$ (self adjoint)

$T, T^{\dagger}$: $\langle T\vec{v}, \vec{w}\rangle = \langle \vec{v}, T^{\dagger}\vec{w}\rangle$
adjoint.

$\underbrace{\forall j, \ \lambda_j \text{ is real}}$ $\quad \mathbb{C}^n$

$$H = \sum_{j=0}^{n-1} \lambda_j |j\rangle\langle j|$$

Unitary: $U^{\dagger} = U^{-1} \iff U^{\dagger}U = \mathbb{1}$

$\forall j, \ |\lambda_j| = 1$

Unitaries preserve length.

$\underbrace{\| U\cdot|a\rangle \|} = \| |a\rangle \|$ $\qquad (A\cdot B)^T = B^T\cdot A^T$

$$(U|a\rangle)^{\dagger}\cdot U|a\rangle = \langle a|\boxed{U^{\dagger}\cdot U}|a\rangle$$
$$= \langle a|\mathbb{1}|a\rangle = \langle a|a\rangle$$
$$= \| |a\rangle \|$$

$$U = \sum_j \lambda_j |j\rangle\langle j|$$

$$U^{\dagger}\cdot U = \sum_{ij} \lambda_j\lambda_i |j\rangle\langle j| \cdot |i\rangle\langle i|$$

$$= \sum_i \underbrace{\lambda_i\cdot\lambda_i} |i\rangle\langle i|$$

$|\lambda_i|^2$

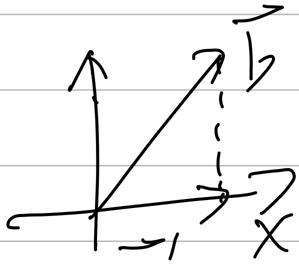
by your self

$$= \sum_{i=0}^{n-1} |i\rangle\langle i| = \mathbb{1}_{n\times n}$$

$$|0\rangle\langle 0| + |1\rangle\langle 1| = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \quad n=2$$

$$\begin{bmatrix}1\\0\end{bmatrix}[1\ 0] \quad \begin{bmatrix}0\\1\end{bmatrix}[0\ 1]$$

Projectors:



$$P = P^2$$

$$P \cdot \vec{b} = \vec{b'} = P(\vec{b'})$$

$$\Rightarrow P = P^2$$

b

Given $P = P^2$ ($\not\Rightarrow$ P is normal)

Math:
- P is Hermitian: orthogonal projector
- P is not ...: oblique projector.

physics. [Nielsen-chuang]:

projector $\Longleftrightarrow$ orthogonal projector.

$$\boxed{\text{normal} \iff \text{Spectral decomposable.}}$$

$\forall j, \ \lambda_j \in \{0, 1\}$

Positive operators M (aka positive semi-definite matrices)

① M is Hermitian

② $\forall j, \ \lambda_j \geq 0$

Positive definite operators (positive difinite matrices)

① M is Hermitian

② $\forall j, \ \lambda_j > 0$