Full-fledged Postulates of (Complex-number) QM

Postulate 1: An isolated quantum system is completed described by its vector of state, which is a unit vector in a Hilbert space.

- vector space (Real ☆complex)
- inner-product space
- complete.

The "mathematically correct" approach to Linear Algebra!!!

$$\vec{u} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \qquad \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\vec{u}^{T} \cdot \vec{u}$$

Modern Math is all about structure.

Mose basic Math structure?

- scale.
- numbers

= $\boxed{\text{Sets}}$

What is a <u>set</u>?

= sets: a collection of things having a common property.

$R = \{ \underbrace{S \mid S \notin S}_{} \}$

$R \overset{?}{\in} R$

Baber only cuts hairs for the

people who don't cut hairs for themselves!

- Sir. Bertrant Russell.

Russell Paradox.

(1901)

"axiomat set theories."

- Zermelo - Fraenkel set theory (ZF)
- Type Theory:

Countability:

① How do you compare the # of even numbers V.S

# of odd numbers ?

② # even numbers v.s.

natural numbers ?

natural even

$n \longmapsto 2n$

③ # natural numbers v.s.

rational numbers ?

④ # Naturals v.s.

Reals ?

"diagnolization"

Gödel's incompleteness theorems

Undecidability.

- Alan Turing,

Magma:

(M, "+")

↑ set ↑ binary operation

1- (Closure)

$$\forall a, b \in M, \quad a + b \in M$$

Semigroup:

(M, "+")

1. (closure)

2. (Associativity)

$\forall a, b, c, \in M,$

$(a+b)+c = a+(b+c)$

Monoid: (M, "+")

① it's a semigroup

3. Identity Element:

$\exists e \in M$ s.t.

$\forall a \in M \quad a+e = e+a = a$

group: (G, "+")

1. Closure: . -- - .

2. Associativity, . . -

3. Identity: $a+e=e+a=a$

4. Inverse element:

$\forall\, a \in G,\ \exists\, -a \in G,$

$a+(-a) = (-a)+a) = e$

5. Commutativity:

$\forall\, a, b \in G$

$a+b = b+a$

(optimal) If satisfied

Abelian Group.

$G = \{ \text{Apple}, \text{Banana} \}$

"A" "B"

$\circledast$

1st Operand	2nd	Result
A $\circledast$	A	B
A	B	A
B	A	A
B	B	B

$$(A \circledast A) \circledast B = A \circledast (A \circledast B)$$

Abelian Group

$\mathbb{Z}_7 = \{0, 1, 2, \ldots, 6\}$

"+": addition mod 7

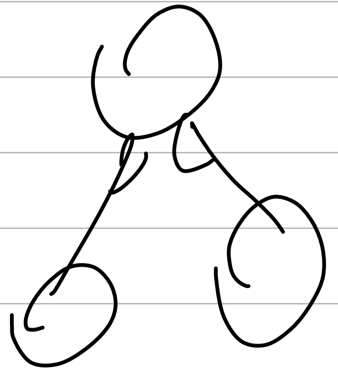
$\boxed{-2}$: by def:

$$2 + \underset{5}{\boxed{-2}} = \overset{e}{0} \quad \underline{\text{mod } 7}$$

Non-abelian Group:

↳ All 2X2 matrices over $\mathbb{R}$ $[\det \neq 0]$ under standard matrix mult.

Groups are important:



H_2O

Ring:

set $(R, "+", "\cdot")$

- R is a abelian group under "+"
- R is a monoid under "$\cdot$":
- $\begin{cases} a \cdot (b+c) = a \cdot b + a \cdot c \\ (b+c) \cdot a = ba + c \cdot a \end{cases}$

distributivity:

"+" ":"

$R = \{a, b, c \ldots\},$

$R[x] \neq \{a, b, c \ldots\} \cup \{x\}$

$\cup \{a + x\}$

$\cup \{a + x + b\}$

$\cup \{ax, ax^2, ax^3, \ldots$
$bx \quad bx^2 \quad \ldots\}$

$ax^2 + bx$

Field: $(\mathbb{F}, +, \cdot)$

1. $\mathbb{F}$ is abelian group "+"

2. $\mathbb{F} \setminus \{\underline{e}\}$ is abelian group under "·"

3. "·" is distributive w.r.t. "+"

$$a\cdot(b+c) = ab+ac$$

$$(b+c)\cdot a = b\cdot a + c\cdot a$$

$$a/b = a\cdot(\underbrace{b^{-1}})$$

$$b\cdot b^{-1} = 1$$

$x + b = c$

$ax^2 + bx + c = 0$

Galoi's theorem
No closed-form formula
for eq. of degree ≥ 5

Modules: ~~R-module~~

$(R, M, +, \cdot)$

2. $(M, +)$ is a abelian group
1.5 R is a ring. under "$\cdot$"
2. $\underset{R}{a} \cdot \underset{R}{b} = \underset{R}{c}$

R m M

$$\cdot : R \times M \longrightarrow M$$

$$\underline{\forall\, r, s \in R, \quad m, n \in M} \ :$$

(a) $\underline{(r+s)} \cdot m = r \cdot m + s \cdot m$

(b) $(r \cdot s) \cdot m = r \cdot (s \cdot m)$

(c) $r \cdot \underline{(m+n)} = r \cdot m + rn$

(d) $\exists\, 1$ (unity)

$$1 \cdot m = m.$$

Vector Space

Let $\mathbb{F}$ be a field.

$\in \{\mathbb{R}, \mathbb{C}\}$

Let V be a set.

If $\exists$ "+", "·" s.t.

$(\mathbb{F}, V, \text{"+"}, \text{"·"})$

form a module, then

$(V, +, \text{"·"})$ is called a

vector space over $\mathbb{F}$.

In a vector space:

- What you can:

span, linear indep.

$u, v \in \bar{V}$

$x \in V \notin$

$\text{span}\{\underbrace{u, \bar{v}}\} = \{a \cdot u + b \cdot \bar{v} \mid a, b \in \mathbb{F}\}$

basis, dimension,

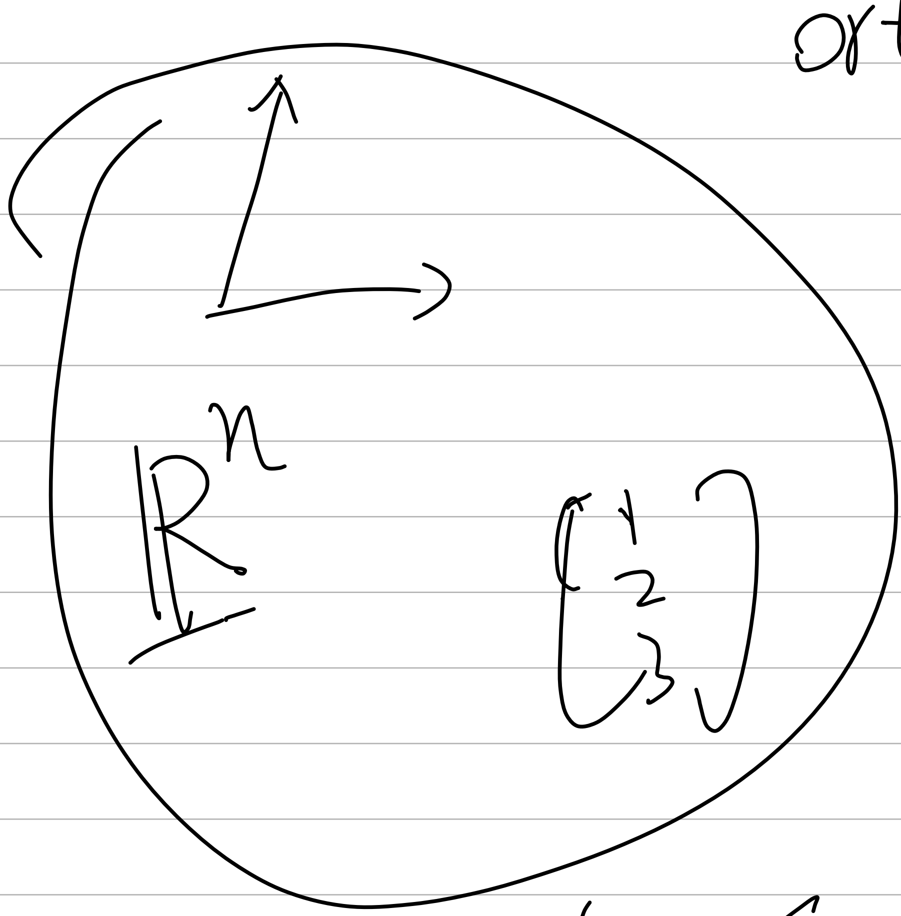
Eigenvalues / vectors.

Linear operator.

- What you cannot:

Any geometry.

length, distance, degree, orthogonality.



Vanilla Vec Space:

(without obvious geometry)

Space $C[a, b]$:

all continuous functions over interval $[a, b] \subseteq \mathbb{R}$

$$f, g \in \underline{\overset{V}{C[a,b]}} \quad a \in \mathbb{F}, \{\mathbb{R}, \mathbb{C}\}$$

$$\begin{cases} (f+g)(x) = f(x) + g(x) \\ (a \cdot f)(x) = a \cdot f(x) \end{cases}$$

$\therefore$ - is a vec space.

Linear Operators:

Let V, W be two vec. spaces (over $\mathbb{F} = \mathbb{C}$)

A linear operator/map/transform

is a function $T: V \to W$

satisfying :

① Additity: $\forall\, u, v \in V$

$$T(u+v) = T(u) + T(v)$$

② Homogeneity:

$\forall\, v \in V,\ a \in \mathbb{F}$

$$T(av) = a \cdot T(v)$$

$$V = \mathrm{span}\{\underline{v_1, \cdots\ v_n}\}$$

$$W = \mathrm{span}\{\underline{w_1, \cdots\cdots\ w_{\underset{\uparrow}{m}}}\}$$

$\forall\, k \in \{1 \ldots n\}.$

$$\underset{\substack{\cap \\ W}}{T(v_k)} = a_{1k} \cdot w_1 + \cdots + a_{mk} w_{mk}$$

$$T = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{m,n} \end{bmatrix}$$

Eigenvalue decomp

L tool to characterize square matrices

$T : V \to W$

of same dim.

$T\vec{v} = \underset{\uparrow}{\lambda} \cdot \vec{v} \quad \lambda \in \mathbb{F}$

<u>Def:</u> (Diagonalizable / Eigenvalue decomp)

a $n \times n$ matrix A over a field $\mathbb{F}$. If $\exists$ invertible matrix P s.t. $\underline{(P^{-1} \cdot A \cdot P)}$ is a diagonal matrix, then A is diagonalizable.

$$\Lambda = P^{-1}AP = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{bmatrix}$$

$$\iff A = \underbrace{P}_{\text{columns are eigenvectors of } A.}\overbrace{\Lambda}^{\lambda_i\text{'s are eigenvalues}}P^{-1}$$

$$A\vec{v} = (P\Lambda P^{-1})\cdot\vec{v}$$

$$= (P^{-1}\cdot\vec{v})$$

$$\uparrow$$

$$\Lambda$$

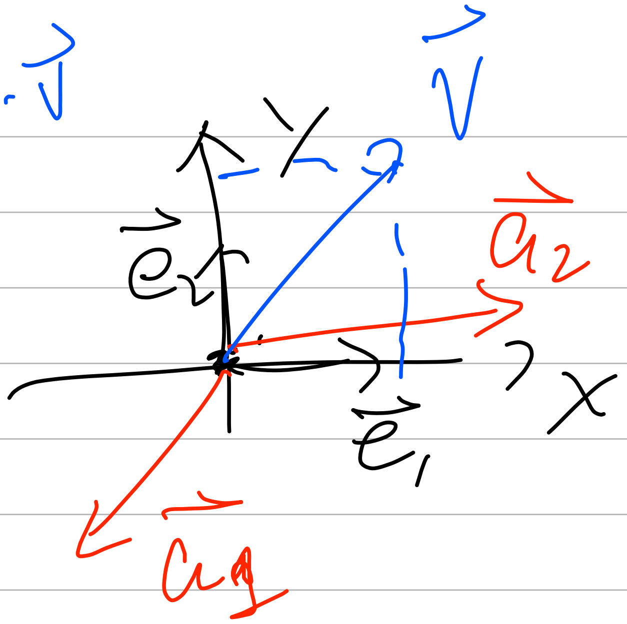
$$\uparrow$$

P → back to origina space

$$\vec{v} \in U = \text{span}\{v_1 \dots v_n\}$$

$A \cdot \vec{v}$

$\mathbb{R}^2$



$$\vec{v} = \alpha_1 \cdot \vec{e_1} + \alpha_2 \cdot \vec{e_2}$$

$$= \underline{\beta_1 \vec{a_1} + \beta_2 \cdot \vec{a_2}}$$

$$(P \cdot \underbrace{P^{-1})\vec{v}}$$

$$A \cdot \vec{v} = \underline{P} \Lambda P^{-1} \vec{v} \qquad \Lambda = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$$

Only one rule to determine if A is

Diagonalizable:

- check # of linearly independent eigenvectors = dim(V)

- linear independence is the most we can say about a diagonalizable A. the eigenvectors

They may not be orthogonal to each other.