CSCI3160 Design and Analysis of Algorithms (2025 Fall)

All-Pairs Shortest Paths

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[1]These slides are primarily based on materials prepared by Prof. Yufei Tao (please refer to Prof. Tao's version from 2024 Fall for the original content). Some modifications have been made to better align with this year's teaching progress, incorporating student feedback, in-class interactions, and my own teaching style and research perspective.

In this lecture, we will study a problem called **all-pairs shortest paths** which is closely related to the SSSP (single-source shortest path) problem discussed in the previous lectures.

Problem Statement

All-Pairs Shortest Paths (APSP)

Input: Let $G = (V, E)$ be a simple directed graph. Let w be a function that maps each edge in E to an integer, which can be positive, 0, **or negative**. It is guaranteed that G has **no negative cycles**.

Output: We want to find a shortest path (SP) from s to t, for all $s, t \in V$. More specifically, the output should be $|V|$ shortest-path trees, each rooted at a distinct vertex in V.

Solutions we already knew:

- If all the weights are non-negative, we can run Dijkstra's algorithm $|V|$ times. The total time is $O(|V|(|V|+|E|)\log|V|)$.
- For the general APSP problem (arbitrary weights), we can run Bellman-Ford's algorithm $|V|$ times. The total time is $O(|V|^2|E|)$.

We will solve the (general) APSP problem in time

$$O\left(|V|(|V|+|E|)\log|V|\}\right).$$

using **Johnson's algorithm**.

Johnson's Algorithm

Recall:

If all the weights are non-negative, we can run Dijkstra's algorithm $|V|$ times. The total running time is $O(|V|(|V| + |E|)\log|V|)$.

We cannot apply Dijkstra's because our graph may have edges with negative weights. Can we convert all the weights into non-negative values **while preserving all shortest paths**?

Interestingly, the answer is **Yes**!

Remark: If the above approach works, doesn't that mean we could use Dijkstra's algorithm to solve the SSSP problem even with negative edge weights? If so, why do we even bother learning Bellman-Ford?

- Answer: the "re-weighting" procedure itself utilizes Bellman-Ford algorithm.

Re-weighting

Fix an arbitrary function $h : V \to \mathbb{Z}$, where $\mathbb{Z}$ denotes the set of integers.

For each edge (u, v) in E, redefine its weight as:

$$w'(u, v) = w(u, v) + h(u) - h(v).$$

Denote by G' the graph where

- the set V of vertices and the set E of edges are the same as G;
- the edges are weighted using function w'.

Re-weighting

Lemma: Consider any path $v_1 \to v_2 \to \ldots \to v_x$ in G where $x \geq 1$. If the path has length ℓ in G, then it has length $\ell + h(v_1) - h(v_x)$ in G'.

Proof: The length of the path in G' is

$$\begin{aligned}
& \sum_{i=1}^{x-1} w'(v_i, v_{i+1}) \\
= & \sum_{i=1}^{x-1} (w(v_i, v_{i+1}) + h(v_i) - h(v_{i+1})) \\
= & \left(\sum_{i=1}^{x-1} w(v_i, v_{i+1}) \right) + h(v_1) - h(v_x).
\end{aligned}$$

$\square$

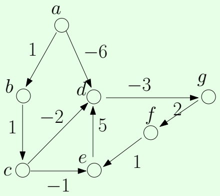
Re-weighting

Corollary: Let π be a shortest path from vertex u to vertex v in G, it is also a shortest path from u to v in G'.

Proof: Let π' be any other path from u to v in G'. Denote by ℓ and ℓ' the length of π and π' in G, respectively. It holds that $\ell \leq \ell'$. By the lemma of the previous slide, we know that π and π' have length $\ell + h(u) - h(v)$ and $\ell' + h(u) - h(v)$ in G', respectively. $\square$

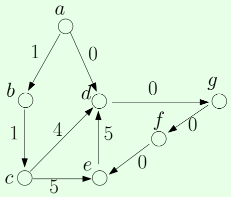
Example

Example:



$h(a) = 0$
$h(b) = 0$
$h(c) = 0$
$h(d) = -6$
$h(e) = -6$
$h(f) = -7$
$h(g) = -9$

After re-weighting, we get graph G':



We want to make sure

$$w'(u, v) \geq 0$$

for all edges (u, v) in E. **Not** every function $h(.)$ fulfills the purpose.

Our main goal:

- **efficiently** find an $h(\cdot)$ such that $w'(u, v) \geq 0$ for all edges (u, v) in E.

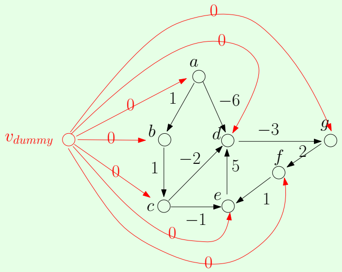
Next, we will introduce a **dummy-vertex trick** to find a good $h(\cdot)$ in $O(|V| \cdot |E|)$ time.

A "Dummy-Vertex" Trick

From $G = (V, E)$, construct a graph $G^\Delta = (V^\Delta, E^\Delta)$ where:

- $V^\Delta = V \cup \{v_{dummy}\}$;
- E^Δ includes all the edges in E, and additionally, a new edge from v_{dummy} to every other vertex in V;
- Each edge inherited from E carries the same weight as in E. Every newly added edge carries the weight 0.

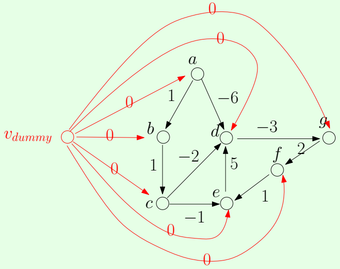
Example:



A “Dummy-Vertex” Trick

In $G^{\Delta} = (V^{\Delta}, E^{\Delta})$, find the shortest path distance from v_{dummy} to every other vertex. This is an SSSP problem which can be solved by Bellman-Ford's algorithm in $O(|V||E|)$ time.

Example:



$spdist(v_{dummy}, a) = 0$
$spdist(v_{dummy}, b) = 0$
$spdist(v_{dummy}, c) = 0$
$spdist(v_{dummy}, d) = -6$
$spdist(v_{dummy}, e) = -6$
$spdist(v_{dummy}, f) = -7$
$spdist(v_{dummy}, g) = -9$

A "Dummy-Vertex" Trick

Recall that we are looking for a good function $h(.)$ to re-weight the edges of G. We now design the function as follows:

$$h(u) = spdist(v_{dummy}, u)$$

for every $u \in V$.

Lemma: After re-weighting the edges of G with the above $h(.)$, all edge weights in G' (i.e., the graph after re-weighting) are non-negative.

The proof is left as an exercise. (Refer to Section 23.3 of [CLRS] for the full proof.)

We can now apply Dijkstra's algorithm to solve the APSP problem in time $O(|V|(|V| + |E|) \log |V|)$.