CSCI3160 Design and Analysis of Algorithms (2025 Fall)

Measuring the Efficiency of an Algorithm by the Worst Input

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[1]These slides are primarily based on materials prepared by Prof. Yufei Tao (please refer to Prof. Tao's version from 2024 Fall for the original content). Some modifications have been made to better align with this year's teaching progress, incorporating student feedback, in-class interactions, and my own teaching style and research perspective.

A significant part of computer science is devoted to understanding the power of the RAM model in solving specific problems, that is, what would be a "fastest" algorithm for each problem.

But how do we measure "fast"? One approach—the one we follow in this course—is to look at the algorithm's cost on the worst input, as we will formalize in this lecture.

Cost on the Worst Input

Define $\mathcal{I}_n$, where n is an integer, to be the set of all inputs to a problem that have the same **problem size** n.

Given an input $I \in \mathcal{I}_n$, the cost $X_{\mathcal{A}}(I)$ of an algorithm $\mathcal{A}$ is the length of its execution on I.

- The **worst-case cost** of $\mathcal{A}$ under the problem size n is the maximum $X_{\mathcal{A}}(I)$ of all $I \in \mathcal{I}_n$.
- The **worst expected cost** of $\mathcal{A}$ under the problem size n is the maximum $\boldsymbol{E}[X_{\mathcal{A}}(I)]$ of all $I \in \mathcal{I}_n$.

Example: Dictionary Search

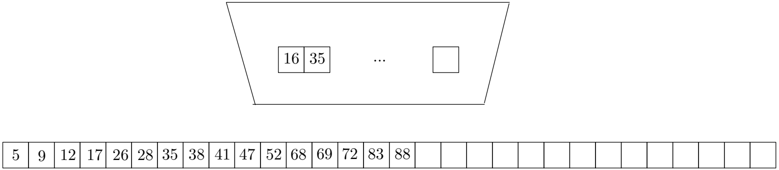
Problem Input: In the memory, a set S of n integers have been arranged in ascending order at the memory cells from address 1 to n. The value of n has been placed in Register 1 of the CPU. Another integer v has been placed in Register 2 of the CPU.

- n is the problem size.
- $\mathcal{I}_n$ is the set of all possible (S, v).

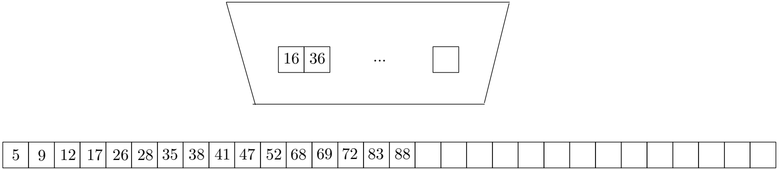
Goal: Determine whether v exists in S.

Example: Dictionary Search

A "yes"-input with $n = 16$



A "no"-input with $n = 16$



Example 1: Dictionary Search

The worst-case cost of the binary search algorithm is $O(\log n)$.

In other words, on any input in $\mathcal{I}_n$, the maximum number $f(n)$ of atomic operations performed by the algorithm grows no faster than $\log_2 n$.

> Note: This does **not** mean $f(n) = \log_2 n$.
> "$f(n) = O(\log n)$" only says that $f(n)$ could be functions like $10(1 + \log_2 n)$, $352 \log_3 n$, $\sqrt{\log n} + 78 \log_2(n^{83})$, etc.

Example 2

Consider the following randomized algorithm:

```
/* A is an array of size n that contains at least one 0 */
do
    r = RANDOM(1, n)
until A[r] = 0
return r
```

What is the expected cost of the algorithm? The answer is "it depends":

- If all numbers in A are 0, the algorithm finishes in $O(1)$ time.
- If A has only one 0, the algorithm finishes in $O(n)$ expected time because
  - $A[r]$ has $1/n$ probability of being 0.
  - In expectation, we need to repeat n times to find the 0. (Think: how to prove this claim formally?)

Example 2 (cont.)

/* A is an array of size n that contains at least one 0 */

1. **do**
2. $r = \text{RANDOM}(1, n)$
3. **until** $A[r] = 0$
4. **return** r

Worst-case cost of the algorithm $= \infty$
Worst expected cost of the algorithm $= O(n)$

We will finish the lecture by tapping into the power of randomization. We will see a problem where randomized algorithms are provably faster than deterministic ones in expected cost.

Before proceeding, think: what is the “expected cost” of a deterministic algorithm?

Power of Randomization

Problem "Find-a-Zero": Let A be an array of n integers, among which half of them are 0. Design an algorithm to report an arbitrary position of A that contains a 0.

For example, suppose $A = (9, 18, 0, 0, 15, 0, 33, 0)$. An algorithm can report 3, 4, 6, or 8.

The Randomized Complexity of "Find-a-Zero"

Power of Randomization

1. **do**
2. $r = \text{RANDOM}(1, n)$
3. **until** $A[r] = 0$
4. **return** r

The algorithm finishes in $O(1)$ expected time on **every input** A!
Think: how to proof this claim formally?

The Classical Complexity of "Find-a-Zero"

In contrast, any deterministic algorithm must probe at least $n/2$ integers of A in the worst case!

Here are two caveats:

- Pay attention to the order of quantifiers: $\exists$ algorithm such that $\forall$ A ...
- Think: how to prove this claim formally? We need to do the following argument: we can treat a deterministic algorithm as making blakc-box queries to the array A, interleaved by some *deterministic* "local" computation steps. So, for any algorithm, you can always construct a "hard" A to enforce a worst-case performance for the given algorithm. (We presented the detailed derivation on the whiteboard. This isn't required for quiz/exam.) Also note that: this proof relies crucially on the order of quantifiers!

In other words, any deterministic algorithm must have a worst case time of $\Theta(n)$—provably slower than the above randomized algorithm ($O(1)$ in expectation).